

Maths (Lecture 9)

Exponential and Logarithmic Functions

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The Exponential Function

- **Exponents revisited**

We have learnt the meaning to be assigned to the expression x^p where x is a positive number. We also know that 'x' is called the base and 'p' is called the exponent (or power).

If m, n are positive integers, there are various cases to consider:

- $a^n = a \times a \times \cdots \times a$ with n terms
- $a^{1/n}$ means the n^{th} root of a . That is, $a^{1/n}$ is that positive number which satisfies
$$(a^{1/n}) \times (a^{1/n}) \times \cdots \times (a^{1/n}) = a$$
where there are n terms on the left hand side.
- $a^{m/n} = (a^{1/n}) \times (a^{1/n}) \times \cdots \times (a^{1/n})$ where there are m terms.
- $a^{-n} = \frac{1}{a^n}$

The basic laws of exponents

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^m)^n = a^{mn}$$
$$a^1 = a, \qquad \text{and if } a \neq 0 \qquad a^0 = 1$$

Example 1

Simplify the expression $\frac{p^{n-2}p^m}{p^3p^{2m}}$

Solution

First we simplify the numerator:

$$p^{n-2}p^m = p^{n-2+m}$$

Next we simplify the denominator:

$$p^3p^{2m} = p^{3+2m}$$

Now we combine them and simplify:

$$\frac{p^{n-2}p^m}{p^3p^{2m}} = \frac{p^{n-2+m}}{p^{3+2m}} = p^{n-2+m}p^{-3-2m} = p^{n-2+m-3-2m} = p^{n-m-5}$$

Exercise

Simplify the expression $\frac{b^{m-n}b^3}{b^{2m}}$

Answer

$$\frac{b^{m+3-n}}{b^{2m}} = b^{m+3-n-2m} = b^{3-m-n}$$

Exponential functions

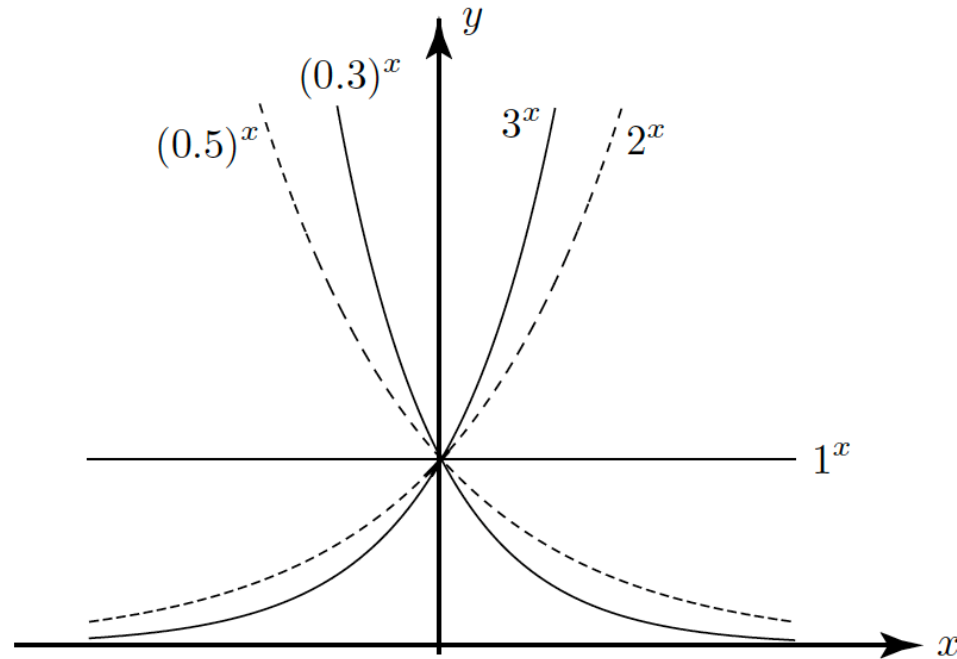
For a fixed value of the base a the expression a^x clearly varies with the value of x : it is a function of x . The function a^x is called exponential function.

If $a > b > 0$ then

$$a^x > b^x \quad \text{if } x > 0$$

and

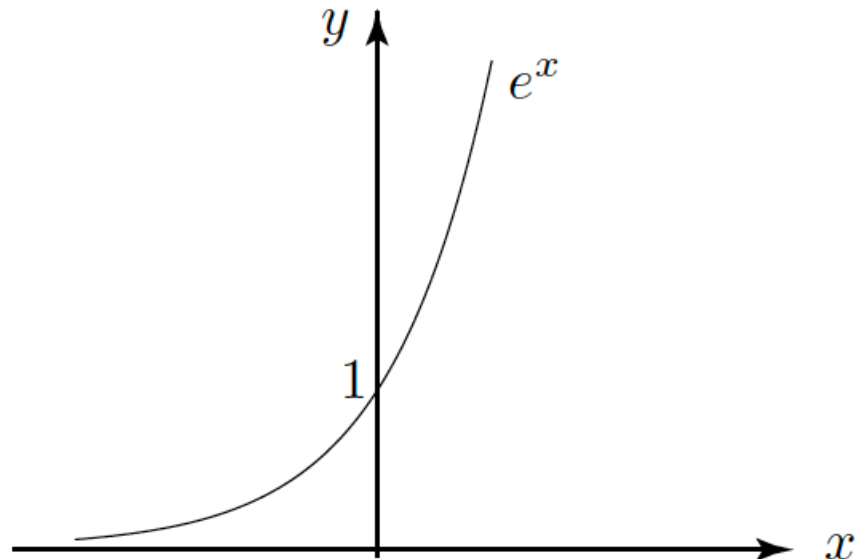
$$a^x < b^x \quad \text{if } x < 0$$



$$e^x$$

The most important and widely used exponential function has the base $e = 2.7182818$. Exponential functions (and variants) appear in various areas of mathematics and engineering.

The function e^{-2x} plays a major role in statistics; it being fundamental in the important normal distribution.



Exponential growth

If $a > 1$ then it can be shown that, no matter how large K is:

$$\frac{a^x}{x^K} \rightarrow \infty \quad \text{as} \quad x \rightarrow \infty$$

The growth of a^x as x increases is called exponential growth.

Example

A function $f(x) = e^{kx}$ grows exponentially and is such that $f(0) = 1$ and $f(2) = 4$. Find the exponential curve that fits through these points, Find the value of k .

Answer

By trying values of k : 0.6, 0.7, 0.8, find the value such that $e^{kx} \cong 4$:

$$e^{2(0.6)} = 3.32 \text{ (too low)} \quad e^{2(0.7)} = 4.055 \text{ (too high)}$$

Now try values of k : $k = 0.67, 0.68, 0.69$:

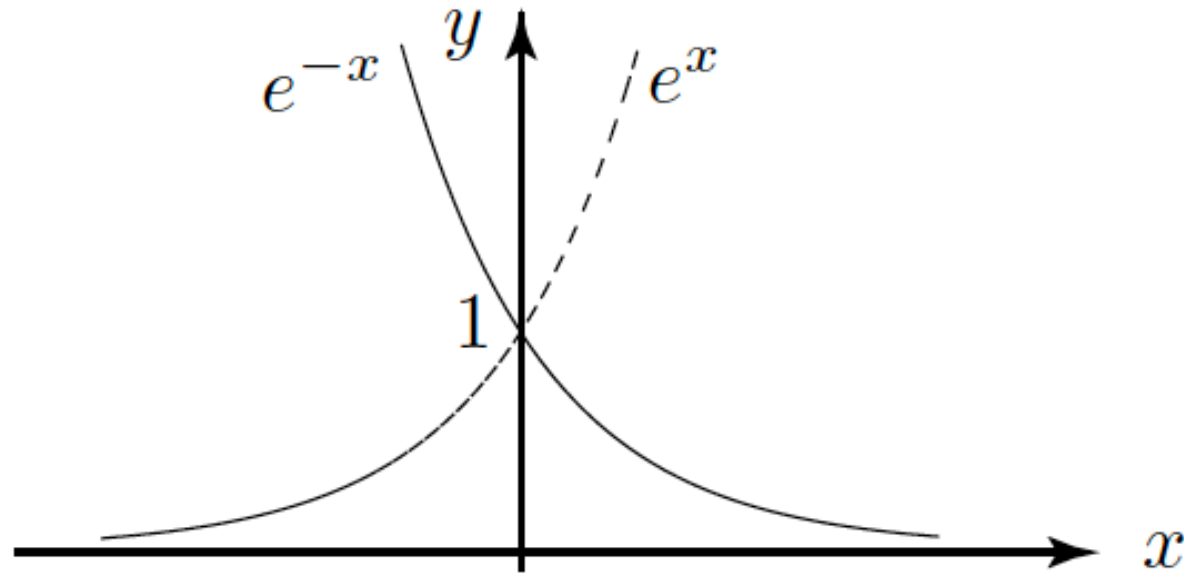
$$e^{2(0.67)} = 3.819 \text{ (low)} \quad e^{2(0.68)} = 3.896 \text{ (low)} \quad e^{2(0.69)} = 3.975 \text{ (low)}$$

Next try values of $k = 0.691, 0.692, 0.693$:

$$e^{2(0.691)} = 3.983, \text{ (low)} \quad e^{2(0.692)} = 3.991 \text{ (low)} \quad e^{2(0.693)} = 3.999 \text{ (low)}$$

Exponential decay

As we have noted, the behaviour of e^x as $x \rightarrow \infty$ is called exponential growth. In a similar manner we characterise the behaviour of the function e^{-x} as $x \rightarrow \infty$ as exponential decay.



Logarithms

Logarithms reverse the process of raising a base 'a' to a power 'n'. As with all exponentials, the base should be a positive number.

If $b = a^n$ then we write $\log_a b = n$.

Of course, the reverse statement is equivalent

If $\log_a b = n$ then $b = a^n$

Example

Determine the log equivalents of

- (a) $16 = 2^4$, (b) $16 = 4^2$, (c) $1000 = 10^3$,
(d) $134.896 = 10^{2.13}$, (e) $8.414867 = e^{2.13}$

Solution

- (a) Since $16 = 2^4$ then $\log_2 16 = 4$
(b) Since $16 = 4^2$ then $\log_4 16 = 2$
(c) Since $1000 = 10^3$ then $\log_{10} 1000 = 3$
(d) Since $134.896 = 10^{2.13}$ then $\log_{10} 134.896 = 2.13$
(e) Since $8.41467 = e^{2.13}$ then $\log_e 8.414867 = 2.13$

Exercise

Find the log equivalent of (a) $b = a^n$, (b) $c = a^m$, (c) $bc = a^{n+m}$

Answer

$$n = \log_a b \qquad m = \log_a c \qquad n + m = \log_a(bc)$$

The laws of logarithms

$$1. \quad \log_a(AB) = \log_a A + \log_a B$$

$$2. \quad \log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$$

$$3. \quad \log_a(A^k) = k \log_a A$$

$$4. \quad \log_a(a^A) = A$$

$$5. \quad \log_a a = 1$$

$$6. \quad \log_a 1 = 0$$

The laws of indices

$$1. \quad a^A a^B = a^{A+B}$$

$$2. \quad a^A / a^B = a^{A-B}$$

$$3. \quad (a^A)^k = a^{kA}$$

$$4. \quad a^{\log_a A} = A$$

$$5. \quad a^1 = a$$

$$6. \quad a^0 = 1$$

Example

Simplify $\log_{10}\left(\frac{1}{10}\right) - \log_{10}\left(\frac{10}{27}\right) + \log_{10} 1000$

$$\log_{10}\left(\frac{1}{10}\right) = \log_{10} 1 - \log_{10} 10 = 0 - 1 = -1$$

$$\log_{10}\left(\frac{10}{27}\right) = \log_{10} 10 - \log_{10} 27 = 1 - \log_{10} 27$$

$$\log_{10} 1000 = 3$$

$$\log_{10}\left(\frac{1}{10}\right) - \log_{10}\left(\frac{10}{27}\right) + \log_{10} 1000 = 1 + \log_{10} 27$$

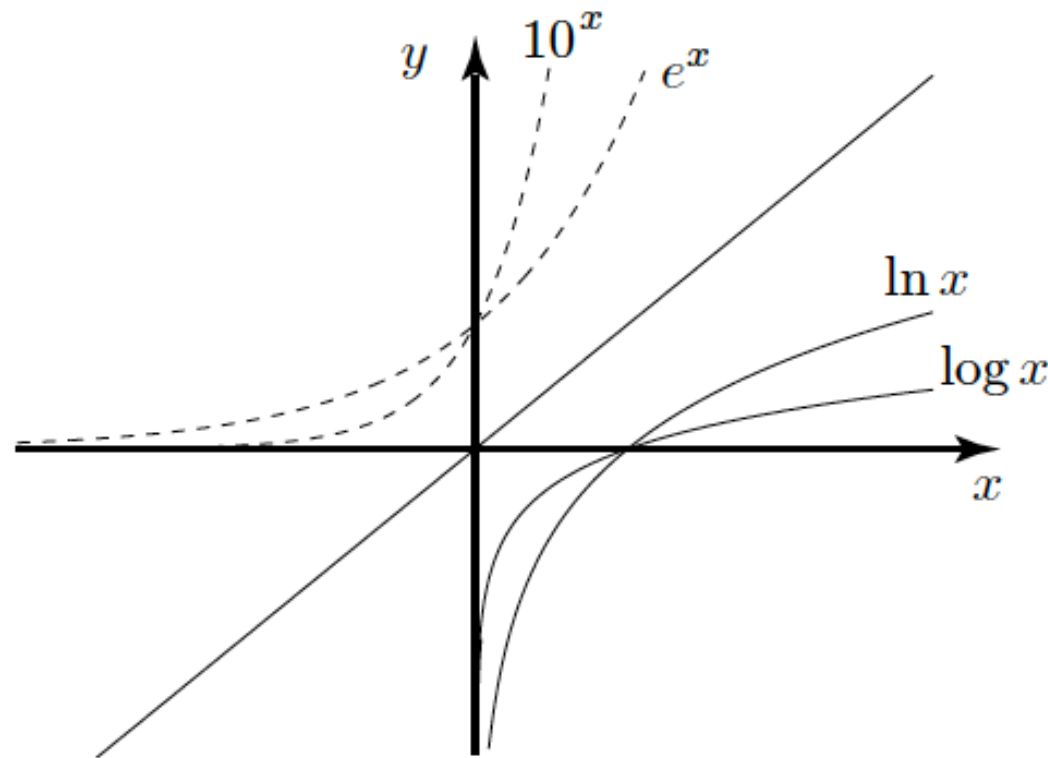
Logs to base 10 and natural logs

In practice only two kinds of logarithms are commonly used, those to base 10, written $\log_{10}$ (or just simply $\log$) and those to base e , written $\log_e$ or more usually $\ln$ (called natural logarithms)

$$\log 13 = 1.11394 \quad (\text{implying } 10^{1.11394} = 13), \quad \ln 23 = 3.13549 \quad (\text{implying } e^{3.13549} = 23)$$

The Logarithmic Function

The functions $y = \log x$ and $y = \ln x$ have similar characteristics. We can never choose x as a negative number since 10^y and e^y are each always positive.



Solving equations involving logarithms and exponentials

To solve equations which involve logarithms or exponentials we need to be aware of the basic laws which govern both of these mathematical concepts.

Example 6

Solve for the variable x : (a) $3 = 10^x$, (b) $10^{x/4} = \log 3$, (c) $\frac{1}{17 - e^x} = 4$

(a) Here we take logs (to base 10 because of the term 10^x) of both sides to get

$$\log 3 = \log 10^x = x \log 10 = x$$

where we have used the general property that $\log_a A^k = k \log_a A$ and the specific property that $\log 10 = 1$. Hence $x = \log 3$ or, in numerical form, $x = 0.47712$ to 5 d.p.

(b) The approach used in (a) is used here. Take logs of both sides: $\log(10^{x/4}) = \log(\log 3)$

$$\text{that is } \frac{x}{4} \log 10 = \log(\log 3) = \log(0.4771212) = -0.3213712$$

So, since $\log 10 = 1$, we have $x = 4(-0.3213712) = -1.28549$ to 5 d.p.

(c) Here we simplify the expression before taking logs.

$$\frac{1}{17 - e^x} = 4 \quad \text{implies} \quad 1 = 4(17 - e^x)$$

or $4e^x = 4(17) - 1 = 67$ so $e^x = 16.75$. Now taking natural logs of both sides (because of the presence of the e^x term) we have:

$$\ln(e^x) = \ln(16.75) = 2.8183983$$

But $\ln(e^x) = x \ln e = x$ and so the solution to $\frac{1}{17 - e^x} = 4$ is $x = 2.81840$ to 5 d.p.

Exercise

Solve the equation $e^{2x} = 17e^x$

Answer.

$$\frac{e^{2x}}{e^x} = 17 \text{ implies } e^{2x-x} = 17 \text{ so } e^x = 17$$

$$x = \ln(17) = 2.8332133$$

Example

Solve $2e^{2x} - 7e^x + 3 = 0$.

First write this equation as a quadratic in the variable $y = e^x$ remembering that $e^{2x} \equiv (e^x)^2$:

$$2y^2 - 7y + 3 = 0$$

$$(2y - 1)(y - 3) = 0 \text{ therefore } y = \frac{1}{2} \text{ or } y = 3$$

$$x = -0.693147 \text{ or } x = 1.0986123$$