

Introduction To MATRICES

A **matrix** is a store for information.

For example, the number of boys and girls in Years 12 and 13 of a college could be recorded as a matrix:

	boys	girls	
Year 12	65	67	or simply $A = \begin{bmatrix} 65 & 67 \\ 68 & 61 \end{bmatrix}$
Year 13	68	61	

A bold capital letter is often used to signify a matrix.

So matrices are essentially stores for **numbers**.

They are rectangular in appearance and are made up of a number of entries or **elements**.

Matrices is the plural of matrix.

A **matrix** is a rectangular array of numbers. Each entry in the matrix is called an **element**.

The order of a matrix

Matrices can come in different sizes. Consider the following:

$$A = \begin{bmatrix} 4 & 2 \\ -2 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 3 & 0 \\ -2 & 1 & 9 \\ 7 & 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 9 & 1 \\ -3 & 0 & 1 \end{bmatrix}$$

Matrices are classified by the number of rows and the number of columns that they have.

In the above:

matrix **A** has 2 rows and 2 columns, it is a 2×2 matrix;

matrix **B** has 3 rows and 3 columns, it is a 3×3 matrix;

matrix **C** has 2 rows and 3 columns, it is a 2×3 matrix.

Read as '2 by 2'

A matrix with m rows and n columns is an $m \times n$ matrix. This is called the **order** of the matrix.

Adding and subtracting matrices

We can add or subtract matrices provided they have the **same order**.

Adding/subtracting matrices is a very simple process since all we do is add/subtract **corresponding elements** from each matrix.

Worked example

The matrices A , B , C and D are given by

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -2 \\ -1 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 9 \\ -4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 \\ -2 \end{bmatrix}.$$

Find the following where possible:

- (a) $A + B$, (b) $A - B$, (c) $C + D$,
(d) $D - C$, (e) $A + C$.

Solution

$$\begin{aligned} \text{(a)} \quad A + B &= \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 7 & -2 \\ -1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3+7 & 6+(-2) \\ 1+(-1) & 3+4 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 0 & 7 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad A - B &= \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 7 & -2 \\ -1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3-7 & 6-(-2) \\ 1-(-1) & 3-4 \end{bmatrix} = \begin{bmatrix} -4 & 8 \\ 2 & -1 \end{bmatrix} \end{aligned}$$

$$\text{(c)} \quad C + D = \begin{bmatrix} 9 \\ -4 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 9+5 \\ -4+(-2) \end{bmatrix} = \begin{bmatrix} 14 \\ -6 \end{bmatrix}$$

$$\text{(d)} \quad D - C = \begin{bmatrix} 5 \\ -2 \end{bmatrix} - \begin{bmatrix} 9 \\ -4 \end{bmatrix} = \begin{bmatrix} 5-9 \\ -2-(-4) \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$\text{(e)} \quad A + C = \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 9 \\ -4 \end{bmatrix}$$

This is impossible to calculate since the matrices have different orders.

Multiples of matrices

Suppose $M = \begin{bmatrix} 3 & -2 \\ 0 & 4 \end{bmatrix}$, then $M + M + M = \begin{bmatrix} 9 & -6 \\ 0 & 12 \end{bmatrix}$.

This is equivalent to $3M$ and each element of matrix M has been multiplied by 3.

It is therefore a simple matter to multiply a matrix by a constant.

For example, $5 \begin{bmatrix} 4 & 1 & 0 \\ -2 & 5 & -9 \end{bmatrix} = \begin{bmatrix} 20 & 5 & 0 \\ -10 & 25 & -45 \end{bmatrix}$.



To multiply a matrix by a constant, simply multiply each element of the matrix by the constant.

Worked example

Given that $A = \begin{bmatrix} 3 & 6 \\ 4 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 \\ 5 & 4 \end{bmatrix}$, find:

- (a) $3A$, (b) $7B$, (c) $4A + 3B$, (d) $5B - 2A$.

Solution

$$\text{(a)} \quad 3A = 3 \begin{bmatrix} 3 & 6 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 9 & 18 \\ 12 & -15 \end{bmatrix}$$

$$\text{(b)} \quad 7B = 7 \begin{bmatrix} -2 & 0 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} -14 & 0 \\ 35 & 28 \end{bmatrix}$$

$$\begin{aligned} \text{(c)} \quad 4A + 3B &= 4 \begin{bmatrix} 3 & 6 \\ 4 & -5 \end{bmatrix} + 3 \begin{bmatrix} -2 & 0 \\ 5 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 12 & 24 \\ 16 & -20 \end{bmatrix} + \begin{bmatrix} -6 & 0 \\ 15 & 12 \end{bmatrix} = \begin{bmatrix} 6 & 24 \\ 31 & -8 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad 5B - 2A &= 5 \begin{bmatrix} -2 & 0 \\ 5 & 4 \end{bmatrix} - 2 \begin{bmatrix} 3 & 6 \\ 4 & -5 \end{bmatrix} \\ &= \begin{bmatrix} -10 & 0 \\ 25 & 20 \end{bmatrix} - \begin{bmatrix} 6 & 12 \\ 8 & -10 \end{bmatrix} = \begin{bmatrix} -16 & -12 \\ 17 & 30 \end{bmatrix} \end{aligned}$$

Multiplying two matrices

You can multiply a row matrix by a column matrix provided they have the same number of elements. You multiply corresponding elements and add the products together.

Hence,

$$\begin{bmatrix} 2 & 5 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = [2 \times 3 + 5 \times 1 + 7 \times 4] = [6 + 5 + 28] \\ = [39]$$

When you multiply two matrices together you are essentially multiplying the rows of the first matrix with the columns of the second.

In order to be able to do this, the number of columns in the first matrix must be the same as the number of rows in the second.



We can multiply two matrices **A** and **B** only if the number of columns of **A** equals the number of rows of **B**.



If **A** is a matrix of order $a \times b$ and **B** is a matrix of order $c \times d$, then the matrix **AB** exists if and only if $b = c$.

The product **AB** will have order $a \times d$.

The method for multiplying two compatible matrices is illustrated in the following examples.

Worked example 4.3

Find the product **AB** when $\mathbf{A} = \begin{bmatrix} 4 & 9 \\ 10 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$.

Solution

Firstly,
order

$$\begin{array}{cc} \mathbf{A} & \mathbf{B} \\ 2 \times 2 & 2 \times 1 \\ \uparrow & \uparrow \\ & \text{the same} \end{array}$$

Check for compatibility

The number of columns of **A** equals the number of rows of **B**, therefore the product matrix exists.

The product matrix will have order 2×1 .

The method for multiplying the two matrices is illustrated step-by-step below:

$$\mathbf{AB} = \begin{bmatrix} 4 & 9 \\ 10 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$

Find the element of the first row:

$$= \begin{bmatrix} (4 \times 5) + (9 \times 8) \\ ? \end{bmatrix} = \begin{bmatrix} 92 \\ ? \end{bmatrix}$$

First row of **A** multiplied by first column of **B**

Find the first element of the second row:

$$= \begin{bmatrix} 92 \\ (10 \times 5) + (3 \times 8) \end{bmatrix} = \begin{bmatrix} 92 \\ 74 \end{bmatrix}$$

Second row of **A** multiplied by the first column of **B**

So the product $\mathbf{AB} = \begin{bmatrix} 92 \\ 74 \end{bmatrix}$.

Worked example

The matrices A , B , C and D are given by

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -2 \\ -1 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 9 \\ -4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 \\ -2 \end{bmatrix}.$$

Find the following where possible:

- (a) $A + B$, (b) $A - B$, (c) $C + D$,
(d) $D - C$, (e) $A + C$.

Solution

$$\begin{aligned} \text{(a)} \quad A + B &= \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 7 & -2 \\ -1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3+7 & 6+(-2) \\ 1+(-1) & 3+4 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 0 & 7 \end{bmatrix} \end{aligned}$$

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$$\text{(e)} \quad A + C = \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 9 \\ -4 \end{bmatrix}$$

This is impossible to calculate since the matrices have different orders.

Multiples of matrices

Suppose $M = \begin{bmatrix} 3 & -2 \\ 0 & 4 \end{bmatrix}$, then $M + M + M = \begin{bmatrix} 9 & -6 \\ 0 & 12 \end{bmatrix}$.

This is equivalent to $3M$ and each element of matrix M has been multiplied by 3.

It is therefore a simple matter to multiply a matrix by a constant.

For example, $5 \begin{bmatrix} 4 & 1 & 0 \\ -2 & 5 & -9 \end{bmatrix} = \begin{bmatrix} 20 & 5 & 0 \\ -10 & 25 & -45 \end{bmatrix}$.



To multiply a matrix by a constant, simply multiply each element of the matrix by the constant.

Worked example

Given that $A = \begin{bmatrix} 3 & 6 \\ 4 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 \\ 5 & 4 \end{bmatrix}$, find:

- (a) $3A$, (b) $7B$, (c) $4A + 3B$, (d) $5B - 2A$.

Solution

$$\text{(a)} \quad 3A = 3 \begin{bmatrix} 3 & 6 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 9 & 18 \\ 12 & -15 \end{bmatrix}$$

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Worked example

Find the product CD when $C = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 9 \\ 6 & 7 \end{bmatrix}$.

Solution

C D
 order 2×2 2×2 $\leftarrow$ Check for compatibility
 $\uparrow$ $\uparrow$
 the same

The number of columns of C equals the number of rows of D , therefore the product matrix exists.

The product matrix will have order 2×2 .

The method for multiplying the two matrices is illustrated step-by-step below:

$$CD = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} 4 & 9 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

Find the first element of the first row:

$$= \begin{bmatrix} (2 \times 4) + (3 \times 6) & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} 26 & ? \\ ? & ? \end{bmatrix}$$

First row of C multiplied by first column of D .

Find the second element of the first row:

$$= \begin{bmatrix} 26 & (2 \times 9) + (3 \times 7) \\ ? & ? \end{bmatrix} = \begin{bmatrix} 26 & 39 \\ ? & ? \end{bmatrix}$$

First row of C multiplied by the second column of D .

Find the first element of the second row:

$$= \begin{bmatrix} 26 & 39 \\ (5 \times 4) + (8 \times 6) & ? \end{bmatrix} = \begin{bmatrix} 26 & 39 \\ 68 & ? \end{bmatrix}$$

Second row of C multiplied by the first column of D .

Find the second element of the second row:

$$= \begin{bmatrix} 26 & 39 \\ 68 & (5 \times 9) + (8 \times 7) \end{bmatrix} = \begin{bmatrix} 26 & 39 \\ 68 & 101 \end{bmatrix}$$

Second row of C multiplied by the second column of D .

So the product $CD = \begin{bmatrix} 26 & 39 \\ 68 & 101 \end{bmatrix}$.

Worked example

Given that $A = \begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 0 \\ -2 & 3 \end{bmatrix}$ find:

(a) AB ,

(b) BA .

Solution

$$(a) AB = \begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 6 \\ -20 & 3 \end{bmatrix}$$

$$(b) BA = \begin{bmatrix} 6 & 0 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 24 & 12 \\ -17 & -1 \end{bmatrix}$$

It should be clear from the examples above that the order in which we multiply two matrices is very important since it makes a difference to the answer.



In general, $AB \neq BA$ (usually).

Matrix multiplication is not, in general, commutative.

In AB , we say that B is pre-multiplied by A .
In BA , we say that B is post-multiplied by A .

Special matrices

The next few sections deal with matrices that have a certain significance or importance.

Square matrices

A **square matrix** is one which has the same number of rows as columns.

A square matrix has order $m \times m$.

A matrix which has the same number of rows and columns is called a **square matrix**.

For example, $\begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix}$, $\begin{bmatrix} 3 & 6 & 0 \\ 1 & -2 & -6 \\ 9 & 2 & 5 \end{bmatrix}$ and

$$\begin{bmatrix} 0 & 2 & 7 & -3 \\ 9 & 5 & 1 & -2 \\ -3 & 4 & 8 & 2 \\ 9 & -3 & 2 & 7 \end{bmatrix}$$

are all square matrices of order 2×2 , 3×3 and 4×4 , respectively.

The zero matrix

Consider the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Notice what happens when we add the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

also $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Similarly, multiplying by this matrix gives

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

also $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

The 2×2 matrix $Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is called the **zero matrix**

since $Z + A = A + Z = A$

and $ZA = AZ = Z$

for any 2×2 matrix A .

So Z leaves any matrix A unchanged under matrix addition, and itself remains unchanged under matrix multiplication. The matrix Z corresponds to the number 0 in 'ordinary' arithmetic.

The identity matrix

Consider the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Notice what happens when we multiply this matrix by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

also $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$



The matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the 2×2 **identity matrix** because when you multiply any 2×2 matrix A by I you get A as the answer.

This means that for any 2×2 matrix A ,

$$IA = AI = A.$$

In other words, multiplication by I (either pre-multiplication or post-multiplication) leaves the elements of A unchanged.



We can actually have identity matrices of any size (as long as they are square matrices).

E.g. the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is called the 3×3 identity matrix.

(You should, of course, notice that you get the matrix A as the answer, irrespective of the order of multiplication.)

The identity matrix in matrix multiplication is rather like the number 1 in ordinary multiplication. (Multiplying any number by 1 leaves a number unchanged.)

Key point summary

- 1 A matrix is a rectangular array of numbers. Each entry in the matrix is called an **element**.
- 2 A matrix with m rows and n columns is an $m \times n$ matrix. This is called the **order** of the matrix.
- 3 We can add or subtract matrices provided they have the **same order**.
- 4 To add/subtract matrices you add/subtract **corresponding elements**.
- 5 To multiply a matrix by a constant, simply multiply each element of the matrix by the constant.
- 6 We can multiply two matrices A and B only if the number of columns of A equals the number of rows of B .
- 7 If A is an $(a \times b)$ matrix and B is a $(c \times d)$ matrix then the product matrix AB exists if and only if $b = c$. The product AB will be of order $a \times d$.
- 8 In general, $AB \neq BA$. Matrix multiplication is not, in general, commutative.
- 9 A matrix which has the same number of rows and columns is called a **square matrix**.

- 10 The matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the 2×2 **identity matrix** because when you multiply any 2×2 matrix A by I you get A as the answer.

This means that for any 2×2 matrix A ,

$$IA = AI = A.$$

- 11 The 2×2 matrix $Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is called the **zero matrix** since $Z + A = A + Z = A$

$$\text{and } ZA = AZ = Z$$

for any 2×2 matrix A .