

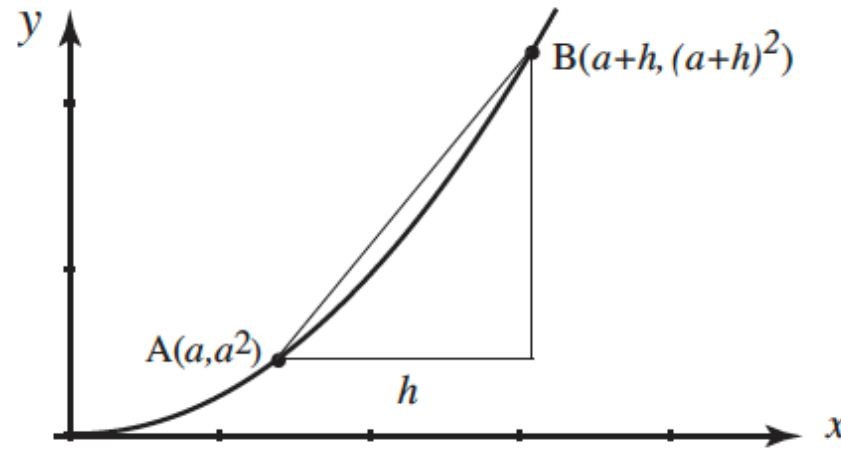
Maths
Lecture 13
(Differentiation)

Finding the gradient at a general point

Consider the graph of $y(x) = x^2$, Let point A be any point with coordinates (a, a^2) .

Let point B be a second point with x coordinate $(a + h)$, y coordinate at B is $(a + h)^2$

.



Therefore the gradient $= \frac{f(B) - f(A)}{h} = \frac{(a+h)^2 - a^2}{h} = \frac{a^2 + 2ha + h^2 - a^2}{h} = 2a + h$

This is the gradient of the line AB. As we let B move closer to A the value of h gets smaller and smaller and eventually tends to zero. We write this as $h \rightarrow 0$.

The gradient of $y = x^2$ is called the derivative of y with respect to x. It has other names too including the rate of change of y with respect to x.

The derivative of $y(x)$ is written $\frac{dy}{dx}$ or $y'(x)$ or simply y'

Differentiating x to the power of something

1) If $y = x^n$, $dy/dx = nx^{n-1}$

2) If $y = kx^n$, $dy/dx = nkx^{n-1}$ (where k is a constant- in other words a number)

Therefore to differentiate x to the power of something you bring the power down to in front of the x, and then reduce the power by one.

Examples

If $y = x^4$, $dy/dx = 4x^3$

If $y = 2x^4$, $dy/dx = 8x^3$

If $y = x^5 + 2x^{-3}$, $dy/dx = 5x^4 - 6x^{-4}$

Table of derivatives

The table lists some of the common functions used in engineering and their corresponding derivatives.

The function in the right-hand column gives the rate of change, or the gradient of the graph, of the function on the left at a particular value of x .

Function	Derivative
constant	0
x	1
kx	k
x^n	nx^{n-1}
kx^n	knx^{n-1}
e^x	e^x
e^{kx}	ke^{kx}
$\ln x$	$1/x$
$\ln kx$	$1/x$
$\sin x$	$\cos x$
$\sin kx$	$k \cos kx$
$\sin(kx + c)$	$k \cos(kx + c)$
$\cos x$	$-\sin x$
$\cos kx$	$-k \sin kx$
$\cos(kx + c)$	$-k \sin(kx + c)$
$\tan x$	$\sec^2 x$
$\tan kx$	$k \sec^2 kx$
$\tan(kx + c)$	$k \sec^2(kx + c)$

Example

Use Table 1 to find $\frac{dy}{dx}$ when y is given by (a) $7x$ (b) 14 (c) $5x^2$ (d) $4x^7$

Solution

- (a) We note that $7x$ is of the form kx where $k = 7$. Using Table 1 we then have $\frac{dy}{dx} = 7$.
- (b) Noting that 14 is a constant we see that $\frac{dy}{dx} = 0$.
- (c) We see that $5x^2$ is of the form kx^n , with $k = 5$ and $n = 2$. The derivative, knx^{n-1} , is then $10x^1$, or more simply, $10x$. So if $y = 5x^2$, then $\frac{dy}{dx} = 10x$.
- (d) We see that $4x^7$ is of the form kx^n , with $k = 4$ and $n = 7$. Hence the derivative, $\frac{dy}{dx}$, is given by $28x^6$.

Example

Use Table 1 to find $\frac{dy}{dx}$ when y is (a) $\sqrt{x}$ (b) $\frac{5}{x^3}$

Answer

(a) Write $\sqrt{x}$ as $x^{\frac{1}{2}}$, and use the result for differentiating x^n with $n = \frac{1}{2}$.

$$\frac{dy}{dx} = nx^{n-1} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}. \text{ This may be written as } \frac{1}{2\sqrt{x}}.$$

(b) Write $\frac{5}{x^3}$ as $5x^{-3}$ and use the result for differentiating kx^n with $k = 5$ and $n = -3$.

$$5(-3)x^{-3-1} = -15x^{-4}$$

Example

Use Table 1 to find

$$(a) \frac{dz}{dt} \text{ given } z = e^t \quad (b) \frac{dp}{dt} \text{ given } p = e^{8t} \quad (c) \frac{dz}{dy} \text{ given } z = e^{-3y}$$

Answer

From Table 1, if $y = e^x$, then $\frac{dy}{dx} = e^x$. Hence if $z = e^t$ then $\frac{dz}{dt} = e^t$.

$$\frac{dp}{dt} = 8e^{8t}$$

$$\frac{dz}{dy} = -3e^{-3y}$$

Exercise

Find the derivative, $\frac{dy}{dx}$, when y is (a) $\sin 2x$ (b) $\cos \frac{x}{2}$ (c) $\tan 5x$

Answer

$$\frac{dy}{dx} = 2 \cos 2x$$

$$\frac{dy}{dx} = -\frac{1}{2} \sin \frac{x}{2}$$

$$\frac{dy}{dx} = 5 \sec^2 5x$$

Exercises

Find the derivatives of the following functions with respect to x :

$$(a) 9x^2 \quad (b) 5 \quad (c) 6x^3 \quad (d) -13x^4$$

Answers

$$(a) 18x \quad (b) 0 \quad (c) 18x^2 \quad (d) -52x^3$$

Find the derivative of each of the following with respect to the appropriate variable:

$$(a) \sin 5x \quad (b) \cos 4t \quad (c) \tan 3r \quad (d) e^{2v} \quad (e) \frac{1}{e^{3t}}$$

Answers

$$(a) 5 \cos 5x \quad (b) -4 \sin 4t \quad (c) 3 \sec^2 3r \quad (d) 2e^{2v} \quad (e) -3e^{-3t}$$

Extending the table of derivatives

The first two rules are for differentiating sums or differences of functions.

Rule 1:	The derivative of $f(x) + g(x)$ is $\frac{df}{dx} + \frac{dg}{dx}$
Rule 2:	The derivative of $f(x) - g(x)$ is $\frac{df}{dx} - \frac{dg}{dx}$

Example

Find the derivative of $y = x^6 + x^4$.

Answer

We simply calculate the sum of the derivatives of each separate function:

$$\frac{dy}{dx} = 6x^5 + 4x^3$$

Example

Find the derivative of $y = 8e^{2x}$

Answer

Here we are interested in differentiating a multiple of the function e^{2x} . We differentiate e^{2x} , giving $2e^{2x}$, and multiply the result by 8. Thus

$$\frac{dy}{dx} = 8 \times 2e^{2x} = 16e^{2x}$$

Example

Find the derivative of $y = 6 \sin 2x + 3x^2 - 5e^{3x}$

Solution

We differentiate each part of the function in turn.

$$\begin{aligned}y &= 6 \sin 2x + 3x^2 - 5e^{3x} \\ \frac{dy}{dx} &= 6(2 \cos 2x) + 3(2x) - 5(3e^{3x}) \\ &= 12 \cos 2x + 6x - 15e^{3x}\end{aligned}$$

Exercise

Find $\frac{dy}{dx}$ where $y = 4 \cos \frac{x}{2} + 17 - 9x^3$.

Answer

First find the derivative of $4 \cos \frac{x}{2}$: $4(-\frac{1}{2} \sin \frac{x}{2}) = -2 \sin \frac{x}{2}$

Then find the derivative of $-9x^3$: $3(-9x^2) = -27x^2$

Finally state the derivative of $y = 4 \cos \frac{x}{2} + 17 - 9x^3$: $-2 \sin \frac{x}{2} - 27x^2$

Evaluating a derivative

Find the value of the derivative of $y = x^3$ where $x = 2$. Interpret your result.

Answer

We have $y = x^3$ and so $\frac{dy}{dx} = 3x^2$.

When $x = 2$, $\frac{dy}{dx} = 3(2)^2 = 12$, that is, $\frac{dy}{dx}(2) = 12$ (Equivalently, $y'(2) = 12$).

The derivative is positive when $x = 2$ and so y is increasing at this point. When $x = 2$, y is increasing at a rate of 12 vertical units per horizontal unit.

Differentiating a product

In this Section we consider how to differentiate non-standard functions - in particular the function that are a product of standard functions.

$$\text{If } y = f(x)g(x) \text{ then } \frac{dy}{dx} = \frac{df}{dx}g(x) + f(x)\frac{dg}{dx} \quad (\text{or } y' = f'g + fg')$$

$$\text{If } y = u.v \text{ then } \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \quad (\text{or } y' = uv' + vu')$$

These versions are equivalent, and called the **product rule**.

Example

Differentiate (a) $y = x^2 \sin x$ (b) $y = x \ln x$

Solution

$$(a) \text{ Here } f(x) = x^2, g(x) = \sin x \quad \therefore \quad \frac{df}{dx} = 2x, \quad \frac{dg}{dx} = \cos x$$

$$\text{and so } \frac{dy}{dx} = 2x(\sin x) + x^2(\cos x) = x(2 \sin x + x \cos x)$$

$$(b) \text{ Here } f(x) = x, g(x) = \ln x \quad \therefore \quad \frac{df}{dx} = 1, \quad \frac{dg}{dx} = \frac{1}{x}$$

$$\text{and so } \frac{dy}{dx} = 1.(\ln x) + x. \left(\frac{1}{x} \right) = \ln x + 1$$

Example

Determine the derivatives of the following functions (a) $y = e^x \ln x$, (b) $y = \frac{e^{2x}}{x^2}$

Answer

$$\frac{dy}{dx} = e^x \ln x + \frac{e^x}{x}$$

$$\frac{dy}{dx} = (-2x^{-3})e^{2x} + x^{-2}(2e^{2x}) = \frac{2e^{2x}}{x^3}(-1 + x)$$

Exercise

In each case find the derivative of the function

(a) $y = x \tan x$

(b) $y = x^4 \ln(2x)$

(c) $y = \sin^2 x$

(d) $y = e^{2x} \cos 3x$

Answer

$$(a) \quad \frac{dy}{dx} = \tan x + x \sec^2 x$$

$$(b) \quad \frac{dy}{dx} = 4x^3 \ln(2x) + \frac{x^4}{x} = x^3(4 \ln(2x) + 1)$$

$$(c) \quad y = \sin x \cdot \sin x$$

$$\therefore \quad \frac{dy}{dx} = \cos x \sin x + \sin x \cos x = 2 \sin x \cos x = \sin 2x$$

$$(d) \quad \frac{dy}{dx} = (2e^{2x}) \cos 3x + e^{2x}(-3 \sin 3x) = e^{2x}(2 \cos 3x - 3 \sin 3x)$$

Differentiating a quotient

In this Section we consider functions of the form $y = \frac{f(x)}{g(x)}$. To find the derivative of such a function we make use of the following Key Point:

$$\begin{array}{llll} \text{If } y = \frac{f(x)}{g(x)} & \text{then} & \frac{dy}{dx} = \frac{g(x)\frac{df}{dx} - \frac{dg}{dx}f(x)}{[g(x)]^2} & (\text{or } y' = \frac{gf' - g'f}{g^2}) \\ \\ \text{If } y = \frac{u}{v} & \text{then} & \frac{dy}{dx} = \frac{v\frac{du}{dx} - \frac{dv}{dx}u}{v^2} & (\text{or } y' = \frac{vu' - v'u}{v^2}) \end{array}$$

Example

Find the derivative of $y = \frac{\ln x}{x}$

Solution

Here $f(x) = \ln x$ and $g(x) = x$

$$\therefore \frac{df}{dx} = \frac{1}{x} \quad \text{and} \quad \frac{dg}{dx} = 1$$

$$\text{Hence } \frac{dy}{dx} = \frac{x \left(\frac{1}{x} \right) - 1(\ln x)}{[x]^2} = \frac{1 - \ln x}{x^2}$$

Example

Find the derivative of $y = \frac{\ln x}{x}$

Solution

Here $f(x) = \ln x$ and $g(x) = x$

$$\therefore \frac{df}{dx} = \frac{1}{x} \quad \text{and} \quad \frac{dg}{dx} = 1$$

$$\text{Hence } \frac{dy}{dx} = \frac{x \left(\frac{1}{x} \right) - 1(\ln x)}{[x]^2} = \frac{1 - \ln x}{x^2}$$

Exercise

Obtain the derivative of $y = \frac{\sin x}{x^2}$ (a) using the formula for differentiating a product and (b) using the formula for differentiating a quotient.

Answer

(a) Write $y = x^{-2} \sin x$ then use the product rule to find $\frac{dy}{dx}$:

$$y = x^{-2} \sin x \quad \therefore \quad \frac{dy}{dx} = (-2x^{-3}) \sin x + x^{-2} \cos x \quad \therefore \quad \frac{dy}{dx} = \frac{-2 \sin x + x \cos x}{x^3}$$

(b) Now use the quotient rule instead to find $\frac{dy}{dx}$:

$$y = \frac{\sin x}{x^2} \quad \therefore \quad \frac{dy}{dx} = \frac{x^2(\cos x) - (2x) \sin x}{(x^2)^2} = \frac{x \cos x - 2 \sin x}{x^3}$$

The Chain Rule

This rule can be used to obtain the derivatives of functions such as e^{x^2+3x} (the exponential function of a polynomial); $\sin(\ln x)$ (the sine function of the natural logarithm function).

When we use a function like $\sin(2x)$ or $e^{\ln x}$ we are in fact dealing with a composite function or **function of a function**.

Example

Specify the functions f , g for the composite functions

(a) $\sin 2x$ (b) $\sqrt{x^2 + 1}$ (c) $e^{\ln x}$

Solution

(a) Here f is the sine function and g is the polynomial $2x$. We often write:

$$f(g) = \sin g \quad \text{and} \quad g(x) = 2x$$

(b) Here $f(g) = \sqrt{g}$ and $g(x) = x^2 + 1$

(c) Here $f(g) = e^g$ and $g(x) = \ln x$

The Chain Rule

The Chain Rule

If $y = f(g(x))$, that is, a function of a function, then

$$\frac{dy}{dx} = \frac{df}{dg} \times \frac{dg}{dx}$$

This is called the **chain rule**.

Example

Find the derivatives of the following composite functions using the chain rule and check the result using other methods

(a) $(2x^2 - 1)^2$ (b) $\ln e^x$

Solution

(a) Here $y = f(g(x))$ where $f(g) = g^2$ and $g(x) = 2x^2 - 1$. Thus

$$\frac{df}{dg} = 2g \quad \text{and} \quad \frac{dg}{dx} = 4x \quad \therefore \quad \frac{dy}{dx} = 2g \cdot (4x) = 2(2x^2 - 1)(4x) = 8x(2x^2 - 1)$$

(b) Here $y = f(g(x))$ where $f(g) = \ln g$ and $g(x) = e^x$. Thus

$$\frac{df}{dg} = \frac{1}{g} \quad \text{and} \quad \frac{dg}{dx} = e^x \quad \therefore \quad \frac{dy}{dx} = \frac{1}{g} \cdot e^x = \frac{1}{e^x} \cdot e^x = 1$$

Power functions

$$f(g) = g^k$$

Thus, using the chain rule: if $y = [g(x)]^k$ then $\frac{dy}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx} = k g^{k-1} \frac{dg}{dx}$.

Example

Find the derivatives of the following power functions

$$(a) \ y = \sin^3 x \quad (b) \ y = (x^2 + 1)^{1/2} \quad (c) \ y = (e^{3x})^7$$

Answer

$$\frac{dy}{dx} = 3(\sin x)^2 \cos x \text{ which we would normally write as } 3 \sin^2 x \cos x$$

$$\frac{dy}{dx} = \frac{1}{2}(x^2 + 1)^{-1/2} 2x = \frac{x}{\sqrt{x^2 + 1}}$$

$$\frac{dy}{dx} = 7(e^{3x})^6 (3e^{3x}) = 21(e^{3x})^7 = 21e^{21x}$$

Exercise

Obtain the derivatives of the following functions:

$$(a) \left(\frac{2x+1}{3x-1} \right)^4 \quad (b) \tan(3x^2+2x) \quad (c) \sin^2(3x^2-1)$$

Answer

$$(a) -\frac{20(2x+1)^3}{(3x-1)^5} \quad (b) 2(3x+1)\sec^2(3x^2+2x)$$

$$(c) 6x \sin(6x^2-2) \text{ (remember } \sin 2x \equiv 2 \sin x \cos x \text{)}$$

MATLAB

Differentiate function

Example

Use Table 1 to find

(a) $\frac{dz}{dt}$ given $z = e^t$ (b) $\frac{dp}{dt}$ given $p = e^{8t}$ (c) $\frac{dz}{dy}$ given $z = e^{-3y}$

```
syms x
diff(exp(x))
diff(exp(8*x))
diff(exp(-3*x))
```

```
>> Diff
```

```
ans =  
exp(x)
```

```
ans =  
8*exp(8*x)
```

```
ans =  
-3*exp(-3*x)
```

Example

In each case find the derivative of the function

(a) $y = x \tan x$

$$\text{Dt_Y1} = \text{diff}(x * \tan(x))$$

(b) $y = x^4 \ln(2x)$

$$\text{Dt_Y2} = \text{diff}(x^4 * \log(2 * x))$$

(c) $y = \sin^2 x$

$$\text{Dt_Y3} = \text{diff}((\sin(x))^2)$$

(d) $y = e^{2x} \cos 3x$

$$\text{Dt_Y4} = \text{diff}(\exp(2 * x) * \cos(3 * x))$$

$$(a) \frac{dy}{dx} = \tan x + x \sec^2 x$$

$$(b) \frac{dy}{dx} = 4x^3 \ln(2x) + \frac{x^4}{x} = x^3(4 \ln(2x) + 1)$$

$$(c) y = \sin x \cdot \sin x$$

$$\therefore \frac{dy}{dx} = \cos x \sin x + \sin x \cos x = 2 \sin x \cos x = \sin 2x$$

$$(d) \frac{dy}{dx} = (2e^{2x}) \cos 3x + e^{2x}(-3 \sin 3x) = e^{2x}(2 \cos 3x - 3 \sin 3x)$$

```
>> Dt_Y2=diff(x^4*log(2*x))
Dt_Y2 =
4*x^3*log(2*x) + x^3
```

```
>> Dt_Y2=diff(x^4*log(2*x))
Dt_Y2 =
4*x^3*log(2*x) + x^3
```

```
Dt_Y3=diff((sin(x))^2)
Dt_Y3 =
2*cos(x)*sin(x)
```

```
>> Dt_Y4=diff(exp(2*x)*cos(3*x))
Dt_Y4 =
2*cos(3*x)*exp(2*x) - 3*sin(3*x)*exp(2*x)
```

Exercise

Obtain the derivatives of the following functions:

(a) $\left(\frac{2x+1}{3x-1}\right)^4$ (b) $\tan(3x^2+2x)$ (c) $\sin^2(3x^2-1)$

Answer

(a) $-\frac{20(2x+1)^3}{(3x-1)^5}$ (b) $2(3x+1)\sec^2(3x^2+2x)$

(c) $6x \sin(6x^2-2)$ (remember $\sin 2x \equiv 2 \sin x \cos x$)