



UNDERSTANDING MATHEMATICS FOR YOUNG CHILDREN

DEREK HAYLOCK AND ANNE D. COCKBURN

A GUIDE FOR TEACHERS OF CHILDREN 3-7



2

UNDERSTANDING NUMBER AND COUNTING

A CHARMING STORY

A 7-year-old child was asked to write a story to go with the subtraction $5 - 3$. This is what the child wrote (child's spelling retained!):

Once upon a time there lived a nubere and he was called 5 and he was very lonely. Won day he saw a nubere 3 and he was lonely to saw he went up to him and sad well you be my frinde. Yes he sad becace I have no frinde to play with. Saw five and three were good frinds and they were happy and the nexed day they went four a pikenik and had a good time. They had cakes and drink and sandwich and biskites and they came home and got into bed with a hot water botel.

This child chooses here to interpret numbers as characters in a story with lives of their own. This is hardly surprising, given children's experience of writing stories in literacy lessons. But, nevertheless, this child's response may prompt the reader to reflect on these questions:

- To what kinds of concrete situations and pictures might a child connect the symbols and names of numbers if they are to show an understanding of number?
- What exactly is a number?

IN THIS CHAPTER

In this chapter we focus first on this question: what exactly is a number? We do this by analysing the network of connections that are involved in understanding number. The middle section of the chapter explores the idea of counting. In the final section we provide the reader with an analysis of how the concept of number is developed from a mathematical perspective.

WHAT IS THREE?

We begin our exploration of what is a number by asking this question. What is three? We asked teachers in one of our groups to close their eyes and to bring to the forefront of their consciousness an image of the number three. They described the image in their minds:

- I can see three dots arranged in a triangle.
- A set of three fruits: an apple, an orange and a banana.
- Three dots in a line.
- A triangle, a shape with three sides.
- The symbol 3.
- Someone holding up three fingers.

This question has been asked of hundreds of teachers and trainees and almost invariably the responses are similar to those above. They usually fall into two categories. Either the image described is a set of three things: dots, sides of a triangle, fingers, sweets, vague unspecified objects or splodges, and so on; or, less frequently, the picture imagined is just the symbol 3.

NUMBERS, NUMERALS AND DIGITS

Clearly the symbol that we use for three (3) is not the number three. As discussed in Chapter 1, the symbol allows us to represent the concept of 'three', to manipulate it and to relate it to other numbers. But the actual symbol that we use to represent three is fairly arbitrary. Other scripts and languages use other symbols and other words, but the same network of connections constituting the concept of three lies behind them. The Roman numeral for three (III), for example, represents the same number as does the symbol 3. If we want to distinguish the symbol from the number it represents then we can refer to the *numeral* 3. So, numerals are the marks we make on paper, such as 3, 26, 819, when we record and manipulate numbers. In practice

we do not often need to make this distinction, because the symbol (that is, the numeral) plays such an important role in representing the number. Another important word that we might mention here is *digit*. There are just ten digits in our number system: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. We use the word 'digit' to refer, for example, to the 8, the 1 and the 9 in the numeral 819.

SETS OF THREE AND ONE-TO-ONE MATCHING

The image of three that clearly dominates the thinking of most adults is the idea that three is a *set* of three things. When a teacher needs to explain to a child a problem involving the number 3 the most likely thing to appear on the table will be a set of three things: three counters, three toys or three fingers, for example. Not surprisingly, therefore, children's own stories about number usually interpret numbers in this way – as sets of things: three children, three teddies, three sweets, three bears, and so on. It could be suggested then that 'three' is what all these sets of three things have in common. In this way, three is seen as a concept abstracted from many examples of sets of three things. One-to-one matching of objects in one set of three with objects in another set of three, as illustrated in Figure 2.1, is an important part of the learning here. It is a visual or practical experience that is designed to focus the child's perception on what is the same about the two sets.

In one-to-one matching, we see the concepts of equivalence and transformation in operation. For example, when a child matches a set of three cups, one to one, onto a set of three saucers, the operation of matching shifts the focus from what is different about the cups and the saucers to what is the same, namely, that there are three of them.

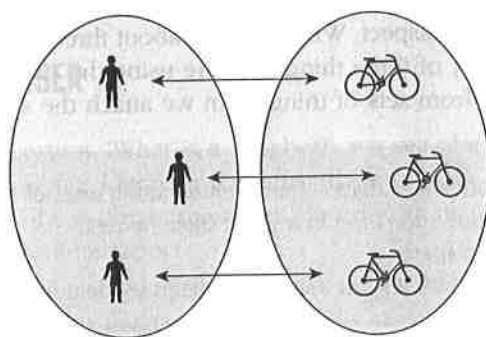


Figure 2.1 One-to-one matching

ADJECTIVE OR NOUN?

The following conversation with some of our Key Stage 1 teachers highlights another important point about understanding number:

- D. Is three an adjective or a noun?
- C. It must be an adjective, because you talk about three things. Three apples, three bananas, and so on. It describes the set.
- E. I think it can be both.
- D. To convince me that it can be a noun you need to give me a sentence beginning with the words 'Three is...'
- C. Three is a number.
- E. Three is less than five.
- A. Three is a factor of twelve.
- D. Three is a prime number.

When three changes from an adjective to a noun like this, it is no longer attached to the sets of three things; it has a life of its own. We can talk about three as though it exists independently of the sets to which we might connect it. We would suggest, therefore, that a key step in the development of the concept of number is this transition from number as an adjective to number as a noun.

NOMINAL, CARDINAL AND ORDINAL

In fact, the view outlined above of the way in which number concepts develop is rather simplistic and does not recognize the complex network of connections between concrete experience, language, pictures and the symbol 3, which constitute the concept of three. So far we have only talked about connecting the symbol with concrete situations consisting of sets of three things. This is only one aspect of a number, referred to as the *cardinal* aspect. When we talk about three bears, three fingers, three children, or any other set of three things, we are using the cardinal aspect of number. But to what else, apart from sets of things, can we attach the symbol 3?

- What about the 3 on a clock face – that's not denoting a set of three things, is it?
- I live at number 3, but I don't live in a set of three houses!
- Or page 3 in a newspaper?
- It's like the number 3 bus. It's just a label that helps us identify one bus from another.

The 3 on a number 3 bus is indeed just a label, a number being used in what is called the *nominal* aspect. This is where a number is used just to label various items and to distinguish the items from one another. The 3 on the number 3 bus enables us to know that this is the bus we want to catch and distinguishes it from all the buses with other numbers. When used like this the symbol is hardly operating as a number at all. For a start, the 3 certainly does not mean that there are three of these buses (even though you may wait for ages and then three arrive at the same time!).

And when we see a number 3 bus coming we do not expect the previous bus to have been a number 2 and the next one to be a number 4. Other examples that are essentially using just the nominal aspect of numbers would be telephone numbers and television channels. Numbers used simply as labels are all around us.

However, behind the other responses from the teachers above is another very important category of images of the number 3. This is called the *ordinal* aspect of number: numbers used not just to label things but also to put them in order. Room number 3, flat number 3, the 3 on the clock face, page 3 in a book – these are labelled 3 because there is some kind of ordering system involved in which they come between 2 and 4. The page before page 3 is invariably page 2 and the page following it is page 4. These are ordinal numbers because they are labelling the pages and telling us where they come in relation to other pages.

Notice that in this ordinal use of the number 3 there is no 'threeness' in the cardinal sense. Page 3 is not a set of three pages: it is, in fact, just one page. It does not have to contain a photograph of a set of three people or a set of three objects, nor does it contain just three words or three sentences. So why is it labelled page 3? The reason is simply that it is located between page 2 and page 4.

We want to emphasize that this ordinal aspect is not an obscure or in any sense a secondary aspect of the concept of number. In spite of the fact that so few people bring to mind an ordinal image when asked to think of three, this ordinal aspect of the number is as central and important as the cardinal aspect. It is one we use frequently in our everyday lives, whether it be locating a page in a book, finding a room in a strange building, looking up a date in a calendar, or deciding whether it's our turn to get padded up ready for the next wicket to fall. We shall see later that in mathematical terms the image of a number line is one that embodies most strongly the ordinal aspect of number.

UNDERSTANDING NUMBER

So, in answer to the question, 'What is a number?', we see already that numbers like three are used as labels for identifying things (the nominal aspect), as labels for putting things in order (the ordinal aspect) and as indications of how many there are in a set of things (the cardinal aspect).

CONNECTING SYMBOLS WITH THE NUMBER LINE

The importance of the ordinal aspect means therefore that a very significant part of understanding number is the connection between the symbols for numbers (the numerals) and the picture of number incorporated in a *number strip* or a *number line*. Examples of these are shown in Figure 2.2. In the number strip, as in many board or playground games, the squares or spaces are labelled with the numerals. In the more sophisticated number line, it is the points rather than the spaces that are labelled. For convenience, the number line in Figure 2.2 is shown as a horizontal line.

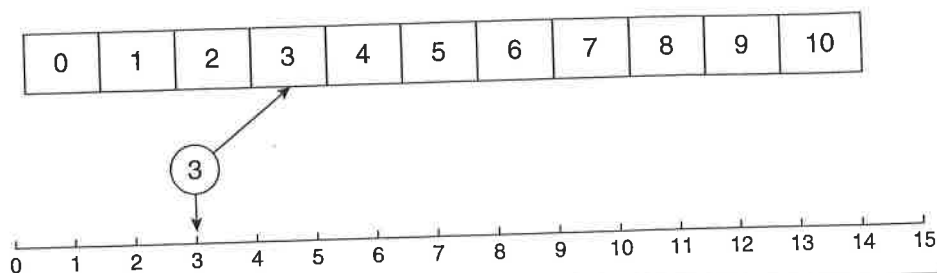


Figure 2.2 *The ordinal aspect of number in a number strip and a number line*



Possibly even more effective for developing number concepts is a similar line arranged vertically, so that the numbers get larger as you go up and smaller as you go down (see photograph). Either of these lines is a picture of number that emphasizes particularly the ordinal aspect. The number three is one point on the number line, or one square on the number strip, not a set of three points or a set of three squares. Its main property is that it lies between two and four.

A NETWORK OF CONNECTIONS

It appears, therefore, that the answer to the question 'What is three?' is far from straightforward. The concept of a number like three appears to involve a network of connections between the symbol 3, the word 'three', concrete situations of sets of three things using the cardinal aspect, and pictures of number involving the ordinal aspect, such as number lines and number strips. Hence it seems that at a very early age the child encounters one enormous difficulty – a difficulty that runs right through mathematics – that one symbol can be used to represent vastly different situations. Not just different in the way in which a set of three sweets is different from a set of three plates, but as different as

the questions, 'How many pages have you read today?' and 'What page are you on?' Both questions might elicit a response of 'three', but with two completely different meanings, one a cardinal number (the number of pages) and the other an ordinal number (the page number).

Of course, the different meanings of the symbol are not totally unrelated and arbitrary, as, for example, two different meanings of some words might be, such as 'well' (in good health) and 'well' (of water). As we shall see, the cardinal and ordinal aspects come together in counting. But the connections are by no means at a simple level of

perception, and the teacher has a major task to help the child to make the connections and to build them into a coherent network.

Nor is it the case that the cardinal and ordinal aspects of number are the whole story. Consider the use of number in the following sentence, overheard when a teacher was talking to a group of children:

'There are three children in class four who are five.'

This demonstrates the way in which we expect even young children to connect number symbols and words to very different situations. In the space of ten simple words the teacher has used numbers in three widely differing ways. The three children constitute a cardinal 3 and class four is an ordinal 4. The five (years old) is an example of number used in a measuring context. The different ways in which numbers are used in measurement will be discussed in Chapter 8 of this book.

LAYING THE FOUNDATIONS FOR LATER EXPERIENCES OF NUMBER

It is important, then, for teachers of young children to lay foundations of experience and networks of connections on to which future experiences of number can be built. In this respect the cardinal aspect of number is a very limited view of what numbers are. Later on, for example, children will meet negative numbers, a source of great mystery for many students of mathematics. But, in fact, it is not difficult to extend an understanding of numbers to make connections with negative numbers if the ordinal aspect and the associated picture of the number line are a strong part of the concept of number. All that is required is to extend the number line the other side of zero and to use appropriate labels for the new points. In a local department store the buttons in the lift are labelled 3, 2, 1 and 0 for the 3rd, 2nd, 1st and ground floors. How mathematically pleasing to note that the button for the basement is labelled -1! This is a straightforward and obvious extension of ordinal number in a real-life context. In fact, some of the teachers we worked with assured us that by extending the ordinal aspect of number below zero, children as young as 5 can handle negative numbers confidently, using appropriate contexts such as number lines, temperatures, page numbers in a book and going up and down in lifts. One teacher reported her experience of using a number line with 5-year-olds, as follows:

I put a vertical number line on the wall with zero and numbers written above it, up to about 10. We did some counting forwards and backwards. Start at 3 and count forward two; where do you get to? That sort of thing. I had marked some points below zero, but not labelled them. I then asked them to start at 1 and count back three. They did this quite happily and one of the girls amazed me by saying that the point we had arrived at was minus two! I don't know where she got this idea from, but the others picked it up immediately and handled the idea with no difficulty. One of the children pointed out that there was the same pattern of numbers above zero as below zero.

OVER-EMPHASIS ON THE CARDINAL ASPECT

We propose that young children should do as much of their number work moving up and down number lines, or similar manifestations of the ordinal aspect, as they do manipulating counters and blocks and counting objects in sets. This is so that they learn to connect the symbol 3 and the word 'three' just as much with the idea of a label for a point or a position as with sets of things. We cannot stress too strongly that numbers are not just ways of representing sets of things.

We have highlighted in Chapter 1 the significance of children experiencing a wide variation in the contexts and structures in which they encounter a mathematical concept or principle, if they are to achieve the deeper understanding that is associated with mastery. We have talked about mathematical concepts as networks of connections between concrete situations, symbols, language and pictures, and highlighted the problem that understanding the concept involves one symbol being connected to a wide range of often very different situations. We have seen that even the basic and seemingly elementary concept of number has this problem built into it, with the same numeral being applied to very different experiences. The discussion about the possible over-emphasis on the cardinal aspect at the expense of the ordinal aspect raises another point of general applicability and great significance to teachers in the Foundation Stage and Key Stage 1. There is a danger that we might continually reinforce just one particular connection in the network of connections constituting a mathematical concept at the expense of equally important or even ultimately more significant connections. If this one connection dominates the child's thinking about the concept, it may be difficult later to build on new experiences that do not readily connect with this part of the network. For example, if the idea that numbers are sets of things is continually reinforced in the early years and is the dominant connection, it is not surprising that negative numbers appear very mysterious when they are met later on. You cannot think of -3 as a set of things! And if you cannot make connections with previous learning then no real understanding but only rote learning can occur.

ASSIMILATION, RESTRUCTURING AND ACCOMMODATION

Learning with understanding progresses most smoothly by what Piaget and his associates called a process of *assimilation* of new ideas into existing networks of connections; however, when material does not connect readily to the existing network, understanding usually requires a *restructuring* of that network in order to *accommodate* the new experiences (Inhelder and Piaget, 1958). Skemp (1971, Chapters 2 and 3) showed how Piaget's ideas were particularly significant in the formation of mathematical concepts and 'schema' – what we call 'networks of connections'. Many children clearly fail to achieve the appropriate restructurings,

particularly in mathematics, and continue with their limited and inadequate networks of connections, relying on memorization of rules and procedures that make little sense to them. It is because of this that we are emphasizing the importance of providing experiences of number early on – particularly activities with the number line – that will provide a basis for later learning with understanding when children encounter numbers other than those used for counting. It is very important that teachers are aware of the nature of the restructurings required as new mathematical experiences are encountered, so that by skilful, directed questioning, they can help children to reorganize their existing networks of connections in order to accommodate the new experiences. In this way children learn with understanding, rather than by rote.

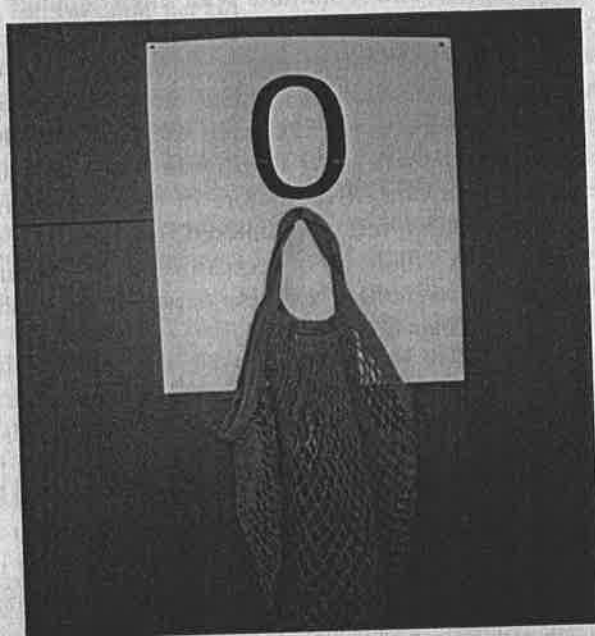
UNDERSTANDING ZERO

We asked the teachers in one of our groups to close their eyes and to think of zero. We asked them to describe the images that came into their minds:

- It's very hard to think of nothing!
- I can see a set with nothing in it, like a circle with nothing inside.
- I thought of putting out three chocolates then eating them!

Is zero a number? Many people feel that it isn't really a proper number; not like one, two or three. It's just nothing. A group of trainees was given a table showing the numbers of children in a class with birthdays in each month and asked which month had the smallest number of birthdays. Some of them had great difficulty in accepting that the month without any birthdays had the smallest number, because they did not think of zero as a number.

Part of the problem with zero is that it's not uncommon for the word 'nothing' to be used for the symbol 0. This fixation on the idea that zero is nothing is, of course, a consequence of the over-emphasis on the cardinal aspect of number. If you have a set of zero objects in your hand then, of course, you do actually have nothing in your hand. But is zero nothing more than nothing? We have already seen that the cardinal view of a number is very limited. Once we consider the ordinal aspect, zero (as a label on the number line, for example) has at least the same status as any other number. In fact, it becomes a very significant and important number. It is the point before 1 on the number line, and sometimes the starting point; it is the point on the number line that separates positive numbers from negative numbers; it is the ground floor in the department store; it is midnight on my digital watch; it is freezing point on the thermometer.



A line of pegs in a classroom is labelled like a number line; there is a bag hanging on each peg, the bag on peg 1 containing one block, that on peg 2 containing two blocks, and so on. The photograph shows the peg labelled 0 and, hanging on this peg, there is an empty bag. This is a bag containing zero blocks. Here we see the two ideas of zero coming together: the position labelled zero (an ordinal zero) and the accompanying bag with nothing in it (a cardinal zero).

So, understanding the concept of zero involves connecting the symbol and the language, not just with 'nothing', an 'absence of something' and empty sets, but also with ordinal pictures of number where zero is very definitely something. Our teachers added some insightful comments on this special number:

- The problem is that we sometimes call it zero, we sometimes call it nought, and sometimes nothing. Other numbers like 1 have only one name.
- When children do take-away questions and get nothing left, they tend to say 'nothing', but when they count backwards they always say 'zero' at the end, like a countdown for a spaceship.
- Counting down is very clearly an ordinal aspect of number, isn't it?
- Do books have a page numbered zero?

Does this book have a page numbered zero? How could you label the pages that come before page 1, other than the way it is done in this book?

PAUSE TO REFLECT ZERO, ZERO EVERYWHERE

Make a list of situations in your everyday life in which you come across the symbol for zero (0). Compare your list with that of a colleague. In each example you come up with, discuss how or why the symbol is being used, what language goes with it, what it means in that context, and any other questions that might occur to you or puzzle you. Try to come up with some examples that are very different in the way the symbol is used.

Here are two examples we came up with:

- 1 All the area telephone codes in Britain begin with a zero (0); for example, Norwich is 01603. Most people read this as 'O' (pronounced 'oh') apart from a few pedants (like Derek) who insist on saying 'zero'. Does this matter? What is this zero for? Does it mean 'nothing' or does it just mean nothing?
- 2 On the formatting palette for a Word document on a computer we spotted: Indentation (cm) Left 0. How do you read this symbol? What does it mean in this context? How is it connected with the picture of a number line on the ruler that can be displayed on the screen? How is it connected with the layout of the text you are typing? Note how you can change the setting in the indentation box up or down: what numbers are available either side of the zero?

UNDERSTANDING COUNTING

For young children their understanding of the concept of number is closely bound up with their experience of counting. Children's first experiences of counting both at home and in the nursery class will include preparing, matching and comparing. Many activities in the early years classroom provide opportunities for children to participate in the preparation of resources, and this often involves counting; for example, to prepare for playing shops the children may count out five pennies (or tokens of some kind) for each child to use to buy items from the play shop. Likewise, there are frequent opportunities for children to match numbers; for example, at snack time they may count ten children and count out ten pieces of banana and then match one piece of banana to each child. Similarly, children will often count to compare, motivated by their strong sense of what is fair; for example, counting to check that there are three children in each group, or to check that each child has three and no more than three pieces of fruit.

My 3- and 4-year-olds know that only three children at a time are allowed to play on the slide in the playground. I see the others keeping an eye on the slide and as soon as the number drops to two they rush to get in as quick as they can. If two arrive at the same time they call me over to sort it out because they realize that there are four children now and that's not allowed!

From a mathematical perspective we may see numbers as describing sets or labelling points, but most significantly for young children, numbers are words you use to count with. In fact, we shall see that it is in the process of counting that the ordinal and cardinal aspects of number come together. So, now we ask, what is counting? What is there to be understood? What connections must be made for a learner to be able to count with meaning? We shall identify a number of key components that are involved in learning to count, drawing substantially on the influential work of Gelman and Gallistel (1986).

PRE-COUNTING EXPERIENCES

It is important to recognize first that learning to count requires a considerable background of experience, much of which many children bring to the Foundation Stage classroom from their life at home. We mention here three kinds of prerequisite experience for counting.

Top of the list is the whole idea of sorting objects into sets and categorization. In order to engage with counting some objects, you clearly need to have the notion of the objects being separated off from those you are not counting. This is an early experience of forming equivalences: identifying a set of objects on the basis of them being in some sense the same: the same colour, the same size, the same family, and so on. Young children's play provides many examples of sorting and categorizing. They will sort blocks, toy animals, model cars, dolls, and so on, into different sets and categories. They categorize when they make distinctions between those in their family and those who are not; between children and adults; between boys and girls. In fact, categorization lies at the heart of language development and is the basic process by which we make sense of our experiences. What we are pointing out here is that this process is the first stage of learning to count: to be able to identify and separate off the members of a set in order that you can then count just these objects and no others.

It's like when you say, 'How many children are here today?' The children in my reception class know they don't count the adults. Or if you ask how many boys, they don't count the girls. When you think about it, that's quite a sophisticated thing for young children to learn to do. There are two steps: they have to separate the children into a category and then count them.

Second, we should not underestimate the importance of the child having a rich experience of talk using language such as 'one more' and 'another one', within the context of family life at home. A basic idea in counting is that numbers increase as you count; that six is more than five, for example. So, when a child learns at such a young age to ask for 'another one', they are getting essential prerequisite experience for learning to count. Parents are probably unaware of the significant experience being provided by the repetition of phrases such as 'one more, then no more'. Teachers, in their own conversations with children, should not be.

Third, there is some evidence to suggest that before they engage in actual counting children are able to distinguish between small numbers, such as one, two and three. Three-year-olds will look at picture books and be able to recognize which picture has one cow, two cows or three cows. So, they are beginning to learn that numbers are used to describe sets of objects and to distinguish between sets of various sizes. This is also experienced when small numbers are used in conversation. Children, before they can count, know that they have two feet, two hands and two eyes, but only one nose. They have stories about three little pigs and three bears. All this experience is invaluable preparation for counting.

Perceiving at a glance the number of items in a set without actually counting them is called *subitizing*. Recognizing a small number like two or three without recourse to any mathematical process or pattern is often called *perceptual* subitizing (Clements, 1999). Children move on from this to *conceptual* subitizing, where it is the pattern in the arrangement of a number of items that leads to its recognition. The arrangement of dots on dice and dominoes (see Figure 7.2 in Chapter 7) and the holes in Numicon plates (see Figure 1.2 in Chapter 1) are therefore important visual images for the development of young children's ability to subitize in a conceptual way.

I find that many of the 3-year-olds arriving in my nursery class can already recognize three items at a glance – and four items if they are arranged in a square. I have one exceptional 3-year-old, who, when she is counting, will spot a group of three things in the set and start from there, counting on 3, 4, 5, 6, ...

THE ORDER OF NUMBERS IS INVARIANT

When children start to learn to count they learn first a pattern of noises, memorized by repetition in all sorts of situations both in and out of school: 'One, two, three, four...' and so on. This set of sounds is probably just as meaningless as many traditional nursery rhymes, demonstrating the young child's amazing capacity for sequential learning. But what they learn from this repetition is the underlying principle of counting that the order of the numbers is fixed. It is invariant: when you are counting, three always comes after two, and so on. Gelman and Gallistel (1986) call this the stable order principle.

ONE-TO-ONE MATCHING OF NUMBER NAMES TO OBJECTS

Then the child has to learn to co-ordinate the utterance of these noises with the physical movements of a finger and the eye along a line of objects, matching one noise to one object. So the child has to learn that one number name must be matched to one object, until all the objects have been used up. Gelman and Gallistel call this the one-one principle. This is a significant skill to learn. Teachers working with children learning to count will know that the two-syllable word 'seven' sometimes poses a difficult problem of co-ordination. Also, children sometimes over count. A way to see if they have a good understanding of the principle is to arrange the objects to be counted in various ways, such as in a straight line, in a circle and in a heap. This will show whether or not the child is aware that they have to keep track of what has been counted and what remains to be counted.

CONNECTING CARDINAL AND ORDINAL ASPECTS

An underlying principle of counting is that the last number that you get to when you are counting a set is the number in the set. Gelman and Gallistel call this the cardinal principle. This is where the ordinal and cardinal aspects come together. As each number is spoken it is being used essentially in an ordinal sense, to label the objects and to order them – number one, number two, number three, and so on. But then the child has somehow to discover that the ordinal number of the last object is the cardinal number of the set. What a stunning and powerful discovery this is! One teacher reported the following:

A 4-year-old girl was counting objects in her head and suddenly announced with great excitement, 'Three is three, isn't it!' She did some more counting in her head and then announced, 'Four is four! And five is five!' It was as though she was counting to three and then realizing that when you got to three you actually had a set of three things. And then that when you got to four you actually had a set of four things, and so on.

COUNTING AS AN ABSTRACTION

A major challenge in learning to do mathematics is to learn to abstract the mathematical concepts from the context in which they are embedded. For example, when a child adds 3 to 4, what actual objects are used makes absolutely no difference to the mathematical process: three sweets and four sweets, three boys and four boys, three counters and four counters, these are all represented by the same abstraction, $3 + 4 = 7$. This is a significant way in which mathematics is so different

from so many of the other things children learn at school. In a poem, a story, a picture, a song, a game, a piece of drama, and so on, the context is the essence of the activity. But in mathematics it is often the case (but by no means always) that the context is of little significance. In counting, children first meet what Gelman and Gallistel call the abstraction principle. It makes no difference what you are counting; whether it be children, animals, counters or fingers, the process is exactly the same. You use the same number names, in the same order, with the same one-to-one matching procedure.

THE ORDER OF THE OBJECTS IS IRRELEVANT

Gelman and Gallistel also identify what they call the order-irrelevance principle. Children have to learn, for example, that whether they count a row of objects from left to right or from right to left makes no difference. In fact, you can count the objects in any order, provided you match them all one-to-one with a number name in the prescribed sequence. This is again a very sophisticated piece of learning and something that teachers should target specifically in the counting experiences of young children and in the assessment of their understanding.

THE ARRANGEMENT OF THE OBJECTS IS IRRELEVANT

An associated principle is that the arrangement of the objects in a set is irrelevant when you are counting them. If a line of seven objects is spread out, there are still seven objects. If they are arranged in a cluster, or a circle, or some other pattern, there are still seven objects. This and the previous principle together constitute what is called the principle of conservation of number. We can understand this in terms of the child learning that whatever transformation is applied to the arrangement of the objects in a set, there is an equivalence that is preserved: there is still the same number. See the Research Focus at the end of this chapter and also Chapter 8 where conservation is discussed more generally in the context of measurement.

MATCHING THE NAMES TO THE NUMERALS

Clearly, at some stage, children have to learn to match the names of numbers to the numerals that represent the numbers. In the early stages of counting they just recite the number names: 'one, two, three, four, five, ...' as they point to objects. At a later stage they will learn to recite these names as they point to the symbols: 1, 2, 3, 4, 5, and so on, doing this often, until eventually the connection between each name and the corresponding numeral is clearly established.

CONNECTING 'ONE MORE' AND THE 'NEXT NUMBER'

There is a basic principle of counting that we should make explicit here, so that teachers reading this book are aware of it and ensure that the principle is targeted in their interactions with children. The principle is simply that the next number after any given number is always one more. So seven is one more than six, because it is the next number after six.

That seems so obvious when you point it out, but it had never occurred to me that it's something children have to learn! I was asking a 4-year-old yesterday what number is one more than five and he couldn't do it. But he can count up to 20 with no trouble!

Again, we suggest this is a sophisticated piece of learning and it is helpful for teachers of young children to be aware of it and to ensure that they make the principle explicit. Even more sophisticated and challenging is the complementary principle: that the previous number is always one less.

Young children find it really hard to think about the previous number. When I ask some of my Year 1 children, 'What comes after five?' they will say 'six', no problem. But if I then ask them what comes before six, they don't have a clue!

THE PATTERN IN COUNTING

Figure 2.3 shows a hundred square. The hundred square is a spatial arrangement of numbers that makes transparent: (a) the ordinal aspect of number; and (b) the relationships associated with place value (see Chapter 3). So, for example, the number 43 is defined by its *position* in the hundred square. What is significant about 43 is that it comes in a row between 42 and 44. This emphasizes the ordinal aspect of number, which, as we have argued above, is often emphasized insufficiently in the Foundation Stage and Key Stage 1, with a consequent over-emphasis on the cardinal aspect. The number 43 here is not a set of things (it is not a cardinal number) – it is a label for one square in an ordered arrangement of squares. The number 43 also refers to the square that comes between 33 and 53 in the column of numbers all ending in the digit 3. This pattern is, of course, significant in terms of children's developing understanding of our base ten place-value system – which is explained in Chapter 3 – and is related to the fact that there are ten squares in each row.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 2.3 *A hundred square*

The hundred square makes clear the patterns in counting up to 100, which are crucial to children's learning to count. First there is the recurring sequence in the last digits. Each sequence of ten numbers ends in 1, 2, 3, 4, 5, 6, 7, 8, 9 and 0. When said out loud the pattern of sounds at the end of the numbers in each row of ten numbers (after 20) is always 'one, two, three, four, five, six, seven, eight, nine, something-ty'. Then there is the pattern in the tens position of the numbers: the way you get a whole row of twenty-somethings, then a whole row of thirty-somethings, then forty-somethings, and so on. Being aware of these patterns in counting and learning to use them are key principles in learning to count beyond twenty.

- One of my 4-year-olds who can count up to 100 emptied the treasure box onto the table, looked at the items and said, 'There's probably more than 100 there.' He uses this phrase 'more than 100' to mean too many for him to count!
- I heard about a kindergarten in Israel that has a clever idea for helping children get a sense of a hundred. They introduced the idea of a 'hundred day celebration'. On the hundredth day at the school the children – either individually or in pairs – are asked to bring in 100 items. They (and their parents!) often prepare for the event with rather grand ideas, but they quickly realize that they would need a very big bag to take in, for example, 100 tennis balls!

It is unfortunate that, in common with most European languages, English does not settle down to a systematic relationship between the language and the symbols until we get beyond the number 19. We would find it much easier to help children to connect the language and the symbols if we counted: '... seven, eight, nine, ten,

onety-one, onety-two, onety-three...' and so on. In this respect children learning about counting and place value in Japan would seem to have a distinct advantage. For example, the names of the numbers 12, 15 and 25 in Japanese (*ju-ni*, *ju-go*, *ni-ju-go*) are all built up from the words for 'ten' (*ju*), 'two' (*ni*) and 'five' (*go*) and could be translated into English as 'ten two', 'ten five' and 'two tens five'. All the number names up to a hundred are constructed in the same logical way, making the place-value principle transparent. By contrast, we might feel sorry for the poor French child who has to learn to connect the symbol for 92, for example, with a name (*quatre-vingt-douze*) that makes no reference to either the 9 tens or the 2 ones, but which could be translated as 'four twenties twelve'!

- The 3- and 4-year-olds in my nursery class always find counting from 11 to 20 the most difficult. Once they've got that sorted they seem to take off and most of them learn the pattern for counting up to 100 very quickly. They love it when they get to 100! It's as though that's the target to aim for and once you get there counting stops!
- Just recently I've noticed that several of the children in my nursery class miss out 15 when they are counting up to 20, which puzzled me. I wonder if it's because when they say the 'four' in 'fourteen' it's the first time they repeat a number they have learnt already; and the pattern of sounds 'four, five, six' that's already fixed in their brains leads them to say 'four-teen, six ...' and so it comes out as 'fourteen, sixteen'.
- I find that children often get to twenty-nine and then want to say twenty-ten. To help them get over this I find myself changing the way I say 29, 39, 49 and so on, when we are counting, slowing down a bit and intoning the 'nine' in a way that suggests something different is coming!

NUMBERS GO ON FOREVER

Once the pattern in counting is clearly established beyond 20 and up to a hundred, children learn a further pattern that enables them to go on into the hundreds, and then the thousands. They begin to get hold of another key idea in counting: that it goes on for ever! There is always the possibility of one more.

Children are always fascinated by the question, what's the biggest number? Then they like it when someone says something like, a million, billion, zillion, or something, and they can say, 'what about a million, billion, zillion and one!'

Teachers should recognize that the child who learns this 'and one' trick has got hold of a key principle of counting.

NOT ALL NUMBERS ARE COUNTING NUMBERS

The final principle to learn about counting is that there is more to the concept of number than counting. Because counting is so strongly linked with our early ideas about numbers and our first experiences of number, we can easily develop a mind-set that numbers are just the words and symbols that we use for counting, in other words, positive, whole numbers. This mistaken, naive notion persists into adult thinking, even though we may have met and used many other kinds of numbers in our lives. So, for example, when some trainee teachers were asked how many numbers there are between 1 and 20, the majority gave the answer 18. They ignored numbers such as 7.8 and $9\frac{1}{2}$, even though all of them had presumably used decimals and fractional numbers for years. Many of these trainees were unable to find two numbers that add up to 9 and differ by 4, simply because when they hear 'number' they automatically think of the numbers we use for counting. In the following section, therefore, we explore the concept of number from a more mathematical perspective. But we hope that we do this in a way that will make clear the challenging and intriguing pedagogical implications of this analysis.

A MATHEMATICAL ANALYSIS OF NUMBER

So, in this section we outline the development of number from a mathematical point of view. Although this introduces some fairly abstract mathematical material, the teachers we have worked with were unanimous that this analysis was both comprehensible and helpful to them in their own understanding of what they are doing in teaching number to young children. It is important for those laying the foundations of number to know what later ideas will have to be built on to those foundations, so that they do not limit children's experiences of number to only those that are appropriate to the numbers we use for counting. Figure 2.4 is a diagrammatic representation of the different kinds of numbers that children will encounter as their experience of mathematics grows.

Moving outwards from the centre of Figure 2.4, we will see that this development of number begins with the positive whole numbers, the numbers we use for counting, called the *natural numbers*. It then proceeds to expand the set of what we understand to be numbers to what are called *integers*. The set of numbers is then expanded further to *rational numbers*, and finally to *real numbers*. Notice that in this sequence each of these sets includes the previous sets.

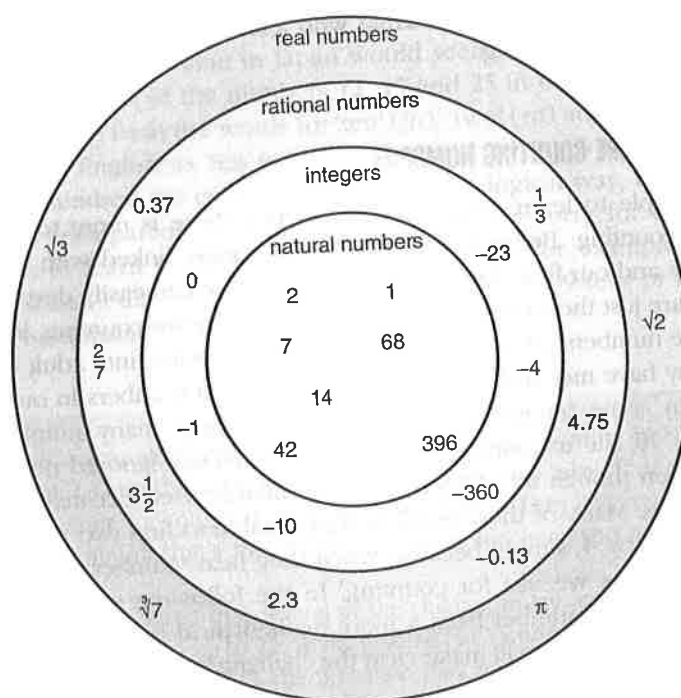


Figure 2.4 A mathematical development of number

UNSTABLE TRUTHS, CHANGING PROPERTIES AND NEW POSSIBILITIES

In the course of the following analysis, we consider each set of numbers in turn. Although the material here is directed at developing the teacher's own understanding of what numbers are, at each stage some pedagogical implications will become apparent. One important theme is that as we move from one set to the next larger set, certain mathematical properties change. Readers who think of mathematics as the subject that offers stability and certainty in its concepts and principles may be somewhat surprised to see it in this light. In fact, as the concept of number is enlarged, some things that were true become no longer true, some things that were false become true and some things that were not possible become possible.

It is important for teachers of young children to be aware of this phenomenon. For example, many children from their early experiences of multiplying with whole numbers (such as 2×3 or 5×2) get the idea that multiplication makes things bigger. This would appear to be true in most cases they encounter – three sets of two objects make a set of six, 5 multiplied by 2 becomes 10, and so on. Perhaps their teachers even encourage them in this notion? But this is a false assertion when you come to multiply by other sorts of numbers. For example, if you multiply 5 by 0.4 the answer (2) is

smaller than the 5 you started with. The problem for us – as teachers – is that these ideas, picked up through early experiences of number, are very persistent and resistant to change. One 15-year-old girl, tackling a complicated algebraic equation, had correctly simplified it to $10 \times x = 5$. Then, having demonstrated impressive mathematical skills in attaining this point, she turned to her teacher and announced that this was stupid because you could not multiply 10 by something and get an answer less than 10! The idea that multiplication makes things bigger – which presumably she had picked up when she was about 7 years of age – had persisted through all those years of subsequent mathematical experience.

A similar misconception is that subtraction makes things smaller. In Chapter 4 we will discuss why $6 - (-3)$ is equal to 9, a case where subtraction clearly makes something larger! Children also get told that you cannot subtract a larger number from a smaller number. If you think that this statement is true yourself, then get a calculator and see what happens when you enter $2 - 6 =$.

THE NATURAL NUMBERS

The set of *natural numbers* consists of those numbers that we use for counting: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, ... going on for ever. We have discussed earlier in this chapter how these numbers have both cardinal and ordinal aspects. The basic properties of these numbers are: (a) that there is a starting number (one); and (b) that for each number there is a *next* number, also called a *successor*. As one of our teachers noted above, this 'nextness' property is an important idea that children find fascinating, particularly with big numbers. They will often ask what is the biggest number in the whole world and will like the idea that no matter how big a number you write down there's always another one following it.

The basic operation that can be performed on this set of natural numbers is *addition*. Any two numbers can be combined using this operation of addition to produce a number, called the *sum* of the two natural numbers, which is still in the set of natural numbers. So, for example, the natural number 12 added to the natural number 14 produces the natural number 26. Mathematicians say that the set is *closed* under addition. In other words, the addition of two natural numbers always produces a natural number. The set of natural numbers together with the basic operation of addition is a closed set. It is because of this that addition with natural numbers does not present us with any challenges to our understanding of how numbers work. It is when we introduce other operations to combine our numbers that problems begin to arise.

INTEGERS

The *inverse* of addition is the operation known as *subtraction*. The concept of inverse is a fundamental mathematical idea that we explain in more detail in Chapter 6.

We call subtraction the inverse of addition because, for example, $6 - 2$ is defined mathematically as meaning what must be added to 2 to give 6. Now, it is soon apparent that the set of natural numbers is not closed under this operation of subtraction. For example, we cannot find a natural number that is equivalent to $2 - 6$. This, of course, is where we get the idea that you cannot subtract a larger number from a smaller one. This is a true statement if you are dealing with natural numbers. However, this does become possible if we extend the set of numbers to include negative numbers and zero. The new set produced by this extension is called the set of *integers*. This set consists of all whole numbers, positive, negative and zero, going on forever in both directions: ... -4, -3, -2, -1, 0, 1, 2, 3, 4 ... Note that the set of integers includes the set of natural numbers.

So, we now take 'number' to mean a member of this set of integers. We then find that subtraction – which was previously not always possible – is now always possible. We can now always subtract one number from another, even when the second number is larger than the first, for example, $2 - 6 = -4$. But other things have changed as well. For a start we have lost the idea of a first number, since now our set of numbers goes on forever to the left of zero as well as to the right. So we could no longer say, for example, that there's nothing less than 1, or even that there's nothing less than zero. We now have a number line extending forever to the left as well as to the right, with points to the left of zero being labelled -1, -2, -3, and so on.

Significantly we should note that we have lost the cardinal aspect of number. We can no longer think of all our numbers as representing sets of things and all our operations on numbers as things that you do with sets of things. We could not demonstrate $2 - 6$, for example, by putting out sets of counters and manipulating them in some way. We could, of course, demonstrate it very simply with a number line, by starting at 2 and counting back 6 – as one of our teachers showed with her 5-year-olds in the example quoted earlier in this chapter. This idea of a number being a set of things, which, as we have seen, so dominates our thinking about number, really does not survive for very long in this mathematical development of the concept of number.

The next operation that appears in this development is *multiplication*. The set of natural numbers is closed under multiplication. In other words, if you multiply one natural number by another, the result, called the *product* of the two numbers, is a natural number, every time. So, for example, the product of 6 and 2 is 12. This is also true of integers. The set of integers is closed under multiplication. We can always multiply any two integers together and find their product. For example, the product of the integers 6 and -2 (think of six steps of -2 on the number line) is the integer -12. The product of the integers -2 and -3 is the integer 6. (The explanation of why mathematicians choose to define the product of -2 and -3 to be the positive integer 6 is abstract and formal and beyond the scope of this book.) So the set of integers is closed under addition, subtraction and multiplication.

RATIONAL NUMBERS

The next step in the development is to think about the inverse of multiplication. This is the operation known as *division*. As with the relationship between subtraction and addition, we call this the inverse because, for example, $6 \div 2$ is defined mathematically as meaning what must be multiplied by 2 to give 6. We find that the set of integers is not closed under this operation of division. For example, we cannot find an integer that is equivalent to $2 \div 6$. Colloquially, we say '6 does not go into 2', meaning there is no integer by which we can multiply 6 to get 2. Similarly, we cannot find an integer equal to $13 \div 5$. We say '5 does not go into 13', meaning there is no integer by which we can multiply 5 to get 13. The only way to deal with this in practical situations where we are restricted to whole numbers is to use the idea of a remainder.

So, from a mathematical perspective, the challenge presented by the fact that the set of integers is not closed under division is to extend the set of numbers to make division always possible. We do this by extending the set of numbers to include *fractions* – and decimals, of course, since decimals, like 0.37 and 4.3, are just fractions, such as $\frac{37}{100}$ and $\frac{43}{10}$, written in a different way. The new set produced by this extension is called the set of *rational numbers*. This set consists of all those numbers that are the *ratio* (hence the word *rational*) of two integers. For example, 2.3 is a rational number because it is the ratio of 23 to 10 (this is also written sometimes as 23:10). It is equivalent to $\frac{23}{10}$ or $23 \div 10$. Similarly, $3\frac{1}{2}$ is a rational number because it is equivalent to $\frac{7}{2}$. And -0.13 is a rational number because it is equivalent to $-\frac{13}{100}$. Note that our set of rational numbers includes the set of integers, which itself included the set of natural numbers. So, for example, the integer -3 is also a rational number because it can be expressed as the ratio $-\frac{3}{1}$, and the natural number 6 is also a rational number because it can be expressed as the ratio $\frac{6}{1}$.

So, we will now take 'number' to mean a member of the set of rational numbers, which includes all whole numbers, positive, negative or zero, and fractions or decimals. One thing we are now able to do is to label points on the number line between the integers. And now we are always able to divide one number by another (with one exception: division by zero is not permissible – see Chapter 6). In other words, the set of rational numbers (excluding zero) is closed under division. We have moved on from the stage where the only ways we can deal with $13 \div 5$ are by saying '5 doesn't go into 13', or by giving the result as '2 remainder 3'. We are now able to recognize an equivalence between $13 \div 5$ and the rational number 2.6, and to record this as $13 \div 5 = 2.6$.

So something that was not always possible is now possible. Other things have changed as well. Suddenly we have lost the 'nextness' property. Numbers no longer have successors. What is the next number after 2.8? Is it 2.9? How about 2.81? or 2.801? Clearly, there is no next number. But an associated gain is that it is now always possible to insert a number between two given numbers. When 'number' meant integer we could not always do this. There was no number between 6 and 7, for example.

But now that 'number' means rational number, this is always possible. And you can go on and on doing this forever.

For example:

- Between 6 and 7 are the numbers 6.5 and 6.6.
- Between 6.5 and 6.6 are 6.63 and 6.64.
- Between 6.63 and 6.64 are 6.638 and 6.639.
- Between 6.638 and 6.639 are ... and so on.

This phenomenon is quite mind-boggling and requires a complete reorganization of our thinking about what numbers are and how they relate to each other. Given any two numbers, it appears that you can go on and on forever putting more and more numbers in between them.

Young children find some difficulty with the concept of *between* when applied to numbers, which is not surprising if the cardinal aspect is dominant in their thinking. 'Between' is a spatial concept. Young children handle it quite successfully in spatial contexts, such as following an instruction to sit between two individuals. This is another case where the number line, which is a spatial image of number, helps to develop an important mathematical concept. For example, using this spatial notion of the concept of between, children can play simple games that involve finding numbers between two given numbers on a number line.

REAL NUMBERS

There is yet another extension to the set of numbers, to what mathematicians call the set of *real numbers*. In the set of real numbers we have all the rational numbers that we have identified already, together with what are called *irrational numbers*. These are numbers that are not rational (that is, not exact fractions or decimals) but which nevertheless still represent real points on the number line, or real lengths. For example, the length of the diagonal of a square whose sides are 1 unit in length is actually equal to the square root of 2 (this is written in symbols as $\sqrt{2}$). This is a real number, representing a real length. It comes on the number line somewhere between 1.4 and 1.5. To be more precise, it comes somewhere between 1.41 and 1.42; or, to be even more precise, somewhere between 1.414213 and 1.414214. But we find that we cannot write down the value of this number exactly in our base ten place-value number system. This means we could not write down a decimal number which, when multiplied by itself, gives the exact answer 2. We could get as close as we like but we could not get exactly 2. There is no rational number that we could put in both of the boxes to make this equation true: $\square \times \square = 2$. All we can say is that $\sqrt{2}$ lies *between* two particular rational numbers. Think of it like this: you can take the bit of the number line between 1 and 2 and divide it into ten equal parts – when you have done this $\sqrt{2}$ will lie between two of the subdivisions, as shown in Figure 2.5. You can then divide this section into ten equal parts and $\sqrt{2}$ will again lie between two of them.

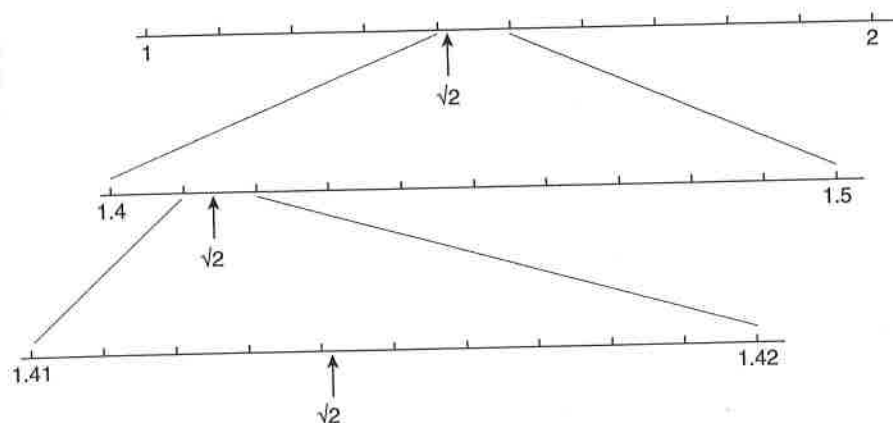


Figure 2.5 *Locating the square root of 2*

And you can go on and on doing this forever and ever, dividing the line into ten equal parts, becoming smaller and smaller, and none of your subdivisions will ever land exactly on $\sqrt{2}$.

Now $\sqrt{2}$ is not a freak. There are millions of numbers like this – in fact an infinite number of them – the square root of 3, the cube root of 7, and so on – all real numbers in the sense that they represent real points on the number line and real lengths. And yet they are not rational numbers; they cannot be expressed exactly as a fraction or a decimal. So we cannot actually write them down exactly in our number system! This is quite a dramatic loss – and a major challenge to our understanding of what numbers are – but what we have gained in the set of real numbers (rational and irrational) is a continuum of numbers, a complete number line in which every point is associated with a number. In the end we get to the idea that the most comprehensive idea of what a number is, is that it is a point on the number line. This notion allows us to embrace all the different kinds of real numbers that we have discussed: positive or negative, whole numbers or fractional numbers, rational or irrational.

THE SIGNIFICANCE OF THIS MATHEMATICAL ANALYSIS

Some readers may have found some of the ideas in this last section difficult to understand. In other words, it may have been difficult to connect them with their existing network of ideas about what numbers are. We hope that this experience might enable us all to appreciate what it is like for children as their experiences of number are gradually broadened. Children learning mathematics have to come to terms with things that were true becoming false, the false becoming true, the impossible becoming possible and even, in our last step, the possible becoming impossible.

Finally, we emphasize that this analysis reinforces the view that, even when young children are only at the stage of handling natural numbers, it is important that their activities with number are not limited to those that are relevant only to natural numbers. When other sorts of numbers are encountered later on, they will then have some framework of experience on which to build. In this chapter we have for this reason made much play, for example, of the importance of number-line experiences. In Chapters 4 and 5, our analysis of number operations, particularly that of subtraction, will provide further illustrations of this important principle.

PAUSE TO REFLECT YOUR NUMBER'S UP!

Here's a way to do some personal research on how numbers are used in everyday experience. Choose a two-digit number that you might expect to hear or see quite often. For example, you might choose the number fifteen, represented by the numeral 15. Look out for every time your number turns up in your life for a week and record the context. Each time you meet it ask yourself, how is this number being used? What does it signify in this context? Is it some kind of a label? Is it being used to put things in order? Is it a number on a scale? In which case, between what other numbers does it sit? Is it telling you how many items there are in a set? Is it a measurement of some kind, attached to units of money, length, weight, time or volume? For example, we have just spotted: page 15 in this book; there's an important conference coming up on 15 July; one of Derek's grandsons is about to have his 15th birthday; a particular breakfast cereal contains 15 grams of sugar per 100 grams; the weather forecast for today includes a temperature of 15 degrees Celsius ...

Compare your record with that of a colleague who has used a different number. Between you discuss what meanings of your numbers seem to be most common in everyday experience. You may be surprised at how infrequently numbers are used to count the items in a set.

RESEARCH FOCUS TALKING AND LISTENING TO CHILDREN

By way of preparation for the research section in this chapter we read a wide range of material dating from the 1970s to the present day. The recurring theme was not so much about numbers and counting but more about exploring young children's understanding of the world. Of particular note was a re-appraisal of Piaget's classic experiments of the 1950s. Among the most famous of these were the tasks he used to explore young children's understanding of the conservation of number. So, for example, a child might be presented with two rows of six counters and asked whether there were the same in number in each row (see Figure 2.6(a)).

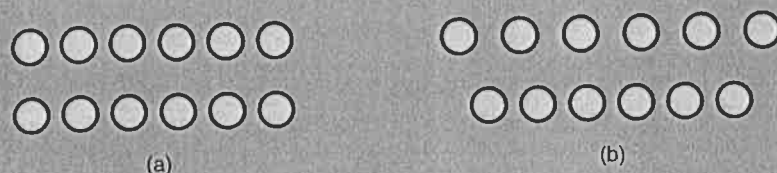


Figure 2.6 *Piaget and Szeminska's number conservation experiment*

Typically, young children replied that the number of counters in each row is the same. Piaget and Szeminska (1952) noted, however, that if one of the rows of counters was stretched out, as in Figure 2.6(b), and the question was repeated, many children under 8 years of age said that now there was not the same number of counters in each row. We discuss conservation further in Chapter 8 (see Figure 8.7). Piaget and Szeminska (1952) concluded that such young children were unable to conserve number because they were unable to appreciate that if you move a group of six objects around there will always be six items in the group.

Let us digress for a moment: if we asked you how many days there are in June you might immediately say '30'. What would your reaction be, however, if we repeated the question immediately after your response? Most probably – we predict – you would wonder whether there really are 30 days in June and whether you might have made a mistake. Reflecting on the possibility that young children might question their initial response if posed the same question a second time, McGarrigle and Donaldson (1974) devised a series of ingenious experiments involving a 'naughty' teddy. Thus, for example, children might have been presented with two sets of counters, as in Figure 2.6(a). A naughty teddy then intervened and 'accidentally' re-arranged one row of counters, as the researcher exclaimed something along the lines of, 'Oh no, he is spoiling our game!' When the children were then asked whether the rows in the new arrangement (as in Figure 2.6(b), for example) each had the same number of counters, they were far more likely to answer correctly. Indeed McGarrigle and Donaldson (1974, p. 347) concluded: 'Most of these four- and five-year olds achieved the criterion for successful conservation of ... number when the transformation was accidental, and yet most of the same children failed when the transformation was effected in the traditional manner.'

Even those who have considerable experience of working with young children sometimes find it hard to fathom what their responses really mean. In their seminal book, *The Child's Understanding of Number*, Gelman and Gallistel (1986, p. 226) reflected that: 'Those who attempt to question children of preschool age directly about their numerical knowledge generally find that the children are not very responsive, nor do their answers seem to indicate much numerical knowledge'. Almost 20 years later another Gelman concluded, 'Sometimes, though, it is hard to know what children mean by the number words they use' (Sarnecka and Gelman, 2004, p. 329).

More recently, the experiments of Nguyen, Kemp and Want (2011) with 84 children in grades 1 and 5 (ages 6–7 years and 10–11 years, respectively) revealed that 'making children's

(Continued)

(Continued)

cognitive tasks funnier can enhance task performance' (p. 154). Cheeseman (2015) stresses the importance of asking challenging questions, listening closely to the responses and, when necessary, reframing a child's answer in order to develop more of a shared understanding of their mathematical thinking. Her research work in Australian classrooms is complemented by that of Jones and Twani (2014) who discuss how 'real conversations' can be initiated within the context of a busy early years setting in the UK.

So what can we conclude? First, although we very much enjoy talking to young children, we can never be entirely confident as to the full extent of their understanding – and we may often underestimate this. Second, to enable an individual to perform at their best – be they young or old – it is a good idea to employ strategies that will engage them. Third, while our approach may seem logical and clear to us it may not to someone else, so it is wise to deploy a series of tasks to explore and evaluate another person's knowledge and understanding. And, finally, as a consequence of the above, we recommend that you read accounts and interpretations of children's work (including those of Haylock and Cockburn) with caution!

SOME ACTIVITIES TO USE WITH CHILDREN

Below are some examples of activities that might be used with children in the lower primary and nursery years to develop their understanding of both cardinal and ordinal number. The emphasis is on understanding. Clearly, activities like these should be complemented by others that focus more directly on counting skills.

ACTIVITY 2.1 SNACK TIME

3-4	4-5	5-6	6-7
☐	☐		

Objective

To develop children's understanding of one-to-one matching and counting sets.

Materials

Beakers, a jug of drink, a plate of pieces of fruit.

Method

Something along these lines can be made into a daily ritual at snack time. The children sit in a circle on the floor. The teacher asks how many children are here today and the class counts them together, with everyone pointing to each child in turn. The teacher gets the

children to count the pieces of fruit to check that it is the same number. Then one child is asked to count out this number of beakers. One child then distributes the beakers, matching one beaker for each child. Another child distributes the fruit similarly. The children are encouraged to count out loud as the items are distributed.



ACTIVITY 2.2 NUMBER DISPLAYS

3-4	4-5	5-6	6-7
☐	☐	☐	☐

Objective

To connect together the language, symbols, concrete experiences and pictures of natural numbers.

Materials

Anything appropriate that the children can find at school or at home!

(Continued)

(Continued)

Method

Make a class display of each natural number from 1 to 10, to which children can add examples from time to time as they come across them. Small groups of children can be given particular responsibility for one display and be encouraged always to be on the lookout for examples of their special number. For example, depending on the age of the children, the display for 4 might contain: pictures of four people cut from a magazine; pictures of other sets of four things, such as four legs on a table or four legs on an animal; some examples of the written word 'four' cut from newspapers; four plastic pennies, labelled 4p; non-English words for 'four'; a Roman numeral IV; a birthday card for a 4-year-old; a house number 4; a square; a photograph of a footballer with 4 on his shirt; a number line with 4 highlighted; page 4 from an old book; a clock showing 4 o'clock; a 4 o'clock programme cut from the television schedules; facts about 4 such as $4 = 3 + 1$, or $4 = 7 - 3$.

Home-based element

Ask the families to be on the lookout with their children for examples of their special number to add to the display, and encourage them to help the children at home to find examples. This could include taking photographs to bring or send to school.

ACTIVITY 2.3 BETWEEN

3-4	4-5	5-6	6-7
	□	□	

Objective

To develop the concept of 'between' in the context of ordinal numbers.

Materials

A blank number strip with spaces for the numerals 0 to 20; a set of cards labelled 0 to 20 to fit in these spaces.

Method

This is a very simple game for three or four children. One child, who is confident in using the number line, acts as umpire. The teacher first shows the children where all the cards from 0 to 20 would go on the number strip, ensuring they understand that it starts at 0 and finishes at 20. The cards are then shuffled and the pack placed face down on the table. The umpire turns over two cards and positions them in their places on the number strip. In the example shown in Figure 2.7 the cards 3 and 12 have been turned over and positioned. Children now take turns to pick up a card from the pack and show where it would go on

the number strip. If the card goes *between* the two numbers the child retains the card. If not, it is discarded. The umpire checks that they do this correctly. For example, in Figure 2.7 a child who picks up 7 retains the card because 7 is between 3 and 12, but a child who picks up 14 has to discard it. The winner of the round is the child who has the most cards at the end. For younger children, use cards and a number strip from 0 to 10.

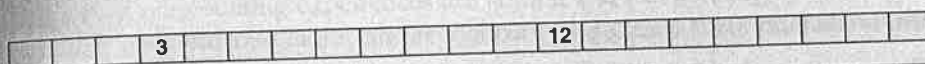


Figure 2.7 *The between game*

ACTIVITY 2.4 I HAVE, WHO HAS? (CARDINAL NUMBERS)

3-4	4-5	5-6	6-7

Objective

To develop recognition of cardinal numbers.

Materials

Prepare a pack of ten cards, with sets of dots on one side and numerals on the other, as shown in Figure 2.8. Note that these are arranged so that the number of dots on one card corresponds to the numeral on the next, in a continuous cycle. For younger children, make a set of cards using numerals and arrangements of dots just from 0 to 5.

Method

This is a game that we use in a number of versions to help children to make various connections in mathematics. Three children play this version of the game. The cards are shuffled and dealt, the remaining card being placed dots upwards in the centre of the table. Each child spreads out their cards in front of them, numerals upwards. The child who has the card with the numeral corresponding to the number of dots displayed places it beside the card in the centre. When all players have agreed that this is correct the card is turned over to reveal the next number to be sought. The first player to dispose of all their cards is the winner. As they play the children should have a conversation along the lines of 'Who has three?' 'I have three ... Who has nine?' and so on.

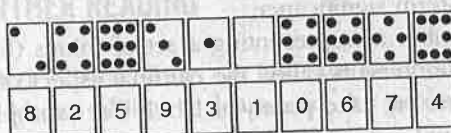


Figure 2.8 *Cards for Activity 2.4*

ACTIVITY 2.5 GREEN BOTTLES

3-4	4-5	5-6	6-7
	□		

Objective

To develop connections between the cardinal and ordinal aspects of number.

Materials

Three number strips labelled from 0 to 10; a pack of 22 cards, two each depicting ten green bottles, nine green bottles, eight green bottles, ... one green bottle, blank.

Method

This is a game for three children, each of whom has a number strip to work on. The cards are shuffled. One at a time each player turns over a card and places it on the ordinal number corresponding to the cardinal number of the set of green bottles shown. A player who at any stage turns over a number already covered on their strip returns the card to the pack. The first player to get three consecutive numbers covered, as, for example, shown in Figure 2.9, wins the round.

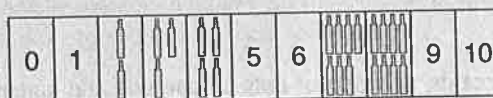


Figure 2.9 Relating cardinal and ordinal numbers in Activity 2.5

SUMMARY OF KEY IDEAS

1. The concept of a number involves a network of connections between the number symbol (the numeral), the number name, pictures (such as the number line) and concrete situations (such as sets of counters).
2. There is a potential danger in continually reinforcing just one particular link in the network of connections constituting a mathematical concept (such as numbers as being sets of things) at the expense of equally important connections (such as numbers as points on a number line), particularly if the connection emphasized is of limited long-term significance.
3. The idea of a number as representing a set of things (for example, three ducks, five fingers, ten counters) is called the cardinal aspect of number.
4. Sometimes numbers are used purely as labels (for example, a number 9 bus). This is called the nominal aspect of number.
5. The idea of number as a label for putting things in order (for example, page 9, room 9) is called the ordinal aspect of number.

6. The
noti
nun
7. Zer
thin
poi
8. Imp
'anc
(sub
9. Key
obje
coun
obje
the
sym
enal
10. Chil
coun
11. The
4, 5,
12. The
-2, -
the s
13. The
ratio
inclu
14. A re
real
(the
tem,
15. As t
ratio
num
prop

SUGGES

Chapters
knowled
coming t
The ex
Nunes ar

6. The ordinal aspect of number, particularly as pictured in the number line, is the notion onto which we can more readily build extensions to our understanding of number, such as negative numbers.
7. Zero does not represent just an empty set (that is, nothing or an absence of something), but also has, for example, important ordinal meanings, such as a significant point on the number line or freezing point in a temperature scale.
8. Important pre-counting experiences are sorting and categorizing, informal use of 'another one' and 'one more', and recognizing at a glance small numbers of items (subitizing).
9. Key principles in learning to count are: the numbers are in a fixed order; one object is matched to one number name; the ordinal number of the last number counted is the cardinal number of the set; counting is an abstraction, so the actual objects counted are irrelevant to the process; the order of the objects is irrelevant; the arrangement of the objects is irrelevant; each number name is matched to a symbol (the numeral); the next number is always one more; the patterns in counting enable counting to continue to a hundred and beyond; numbers go on forever.
10. Children also have to learn that there are numbers other than those we use for counting.
11. The set of natural numbers consists of those numbers used for counting: 1, 2, 3, 4, 5, ... going on forever.
12. The set of integers comprises all whole numbers, positive, negative or zero: ... -4, -3, -2, -1, 0, 1, 2, 3, 4 ... going on forever in both directions. The set of integers includes the set of natural numbers.
13. The set of rational numbers consists of all numbers that can be expressed as the ratio of two integers (that is, all the integers themselves plus all types of fractions, including decimal fractions).
14. A real number is any number representing a point on a number line. The set of real numbers includes the rationals. It also includes irrational numbers, like $\sqrt{2}$ (the square root of 2), which cannot be written down exactly in our number system, but that nevertheless represent real points or real lengths.
15. As the concept of a number develops – through the sets of naturals, integers, rationals, reals – the properties of number change. For example, the cardinality of number is lost in moving from naturals to integers, and the nextness (or successor) property is lost in moving from integers to rationals.

SUGGESTIONS FOR FURTHER READING

Chapters 1–3 of Anghileri (2006) draw on the author's extensive and practical knowledge of young children's ability to learn and the processes of counting and coming to know numbers.

The excellent review of research into children's understanding of mathematics by Nunes and Bryant (1996) provides in Chapters 1–3 some fascinating insights into the

early stages of number and counting. Although over 20 years old this book is still widely read and has been cited more than 100 times in the last four years.

Chapter 4 of Ryan and Williams (2007) discusses the development of number concepts and provides practical examples to reduce the likelihood of mathematical misconceptions developing.

Well worth reading are Sections 2 and 3 of Thompson (2008); a variety of authors offer a range of interesting insights into young children's beliefs and learning about counting, including the contributions of play and experiences at home.

In Chapter 16 of Moyles (2010), Maulfry Worthington, writing about multi-modality play and children's mark-making in mathematics, provides some fascinating insights into children's early mathematical thinking and their use of graphics to support their developing understanding of number.

Chapter 8 of Gifford (2005) focuses on number and counting, including useful sections on subitizing, counting for a purpose, and the use of number stories, songs and games.

For practical suggestions for games and practical activities for children in the early years of schooling, aimed at developing an understanding of number, counting and many other elementary mathematical concepts, have a look at Hansen (2012).

In Chapter 0 of Cockburn and Littler (2008), entitled 'Zero: understanding an apparently paradoxical number', Anne Cockburn and Paul Parslow-Williams provide suggestions on how this elusive number might best be introduced to young children.

Jill Matthews (2014) has written a thought-provoking chapter on children talking about mathematics in Taylor and Harris (eds), which will be particularly useful for those working with very young children. In Chapter 9 of the same book Andrew Harris presents a very useful table describing the typical errors young children make when learning to count. He includes references to the relevant research literature for readers wishing to study the issue in more depth.

Should you find that some children are having difficulty in understanding number and counting it is worth searching the internet for intervention studies. For example, Dyson and colleagues (2015) discuss how low-achieving kindergarten (reception) children benefited more from a number-fact, rather than number-list, intervention.